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THE CUBIC SZEGŐ EQUATION AND HANKEL OPERATORS

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2000 *Mathematics Subject Classification.* — 35B15, 47B35, 37K15.

Key words and phrases. — Cubic Szegő equation, integrable system, NLS, Hankel operator, spectral analysis.

The authors are grateful to Z.Hani, T. Kappeler, S. Kuksin, V. Peller, A. Pushnitski and N. Tzvetkov for valuable discussions and comments about earlier versions of this work, to L. Baratchart for pointing them references [1] and [3], and to A. Nazarov for references concerning Bateman–type formulae. The first author is supported by ANR ANAE 13-BS01-0010-03.
August 27, 2015

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Abstract. —

This monograph is an expanded version of the preprint [16] It is devoted to the dynamics on Sobolev spaces of the cubic Szegő equation on the circle $\mathbb{S}^1$,

$$i\partial_t u = \Pi(|u|^2 u) \ .$$

Here Π denotes the orthogonal projector from $L^2(\mathbb{S}^1)$ onto the subspace $L^2_+(\mathbb{S}^1)$ of functions with nonnegative Fourier modes. We construct a nonlinear Fourier transformation on $H^{1/2}(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1)$ allowing to describe explicitly the solutions of this equation with data in $H^{1/2}(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1)$. This explicit description implies almost-periodicity of every solution in $H^{\frac{1}{2}}_+$. Furthermore, it allows to display the following turbulence phenomenon. For a dense G_δ subset of initial data in $C^\infty(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1)$, the solutions tend to infinity in H^s for every $s > \frac{1}{2}$ with super–polynomial growth on some sequence of times, while they go back to their initial data on another sequence of times tending to infinity. This transformation is defined by solving a general inverse spectral problem involving singular values of a Hilbert–Schmidt Hankel operator and of its shifted Hankel operator.

Résumé. — Cette monographie est une version étendue de la prépublication [16].
Elle est consacrée à l'étude de la dynamique, dans les espaces de Sobolev, de l'équation de Szegő cubique sur le cercle $\mathbb{S}^1$,

$$i\partial_t u = \Pi(|u|^2 u) \ ,$$

où Π désigne le projecteur orthogonal de $L^2(\mathbb{S}^1)$ sur le sous-espace $L^2_+(\mathbb{S}^1)$ des fonctions à modes de Fourier positifs ou nuls. On construit une transformée de Fourier non linéaire sur $H^{1/2}(\mathbb{S}^1)\cap L^2_+(\mathbb{S}^1)$ permettant de résoudre explicitement cette équation avec données initiales dans $H^{1/2}(\mathbb{S}^1)\cap L^2_+(\mathbb{S}^1)$. Ces formules explicites entraînent la presque périodicité des solutions dans $H^{\frac12}_+$. Par ailleurs, elles permettent de mettre en évidence le phénomène de turbulence suivant. Pour un G_δ dense de données initiales de $C^\infty(\mathbb{S}^1)\cap L^2_+(\mathbb{S}^1)$, les solutions tendent vers l'infini à vitesse sur–polynomiale en norme $H^s(\mathbb{S}^1)$ pour tout $s>\frac12$ sur une suite de temps, alors qu'elles retournent vers leur donnée initiale sur une autre suite de temps tendant vers l'infini. Cette transformation est définie *via* la résolution d'un problème spectral inverse lié aux valeurs singulières d'un opérateur de Hankel Hilbert-Schmidt et de son opérateur décalé.

CONTENTS

CHAPTER 1

INTRODUCTION

The large time behavior of solutions to Hamiltonian partial differential equations is an important problem in mathematical physics. In the case of finite dimensional Hamiltonian systems, many features of the large time behavior of trajectories are described using the topology of the phase space. For a given infinite dimensional system, several natural phase spaces, with different topologies, can be chosen, and the large time properties may strongly depend on the choice of such topologies. For instance, it is known that the cubic defocusing nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = |u|^2 u$$

posed on a Riemannian manifold M of dimension $d = 1, 2, 3$ with sufficiently uniform properties at infinity, defines a global flow on the Sobolev spaces $H^s(M)$ for every $s \geq 1$. In this case, a typical large time behavior of interest is the boundedness of trajectories. On the energy space $H^1(M)$, the conservation of energy trivially implies that all the trajectories are bounded. On the other hand, the existence of unbounded trajectories in $H^s(M)$ for $s > 1$ as well as bounds of the growth in time of the $H^s(M)$ norms is a long standing problem [6]. As a way to detect and to measure the transfer of energy to small scales, this problem is naturally connected to wave turbulence. In [7] and [44], it has been established that big H^s norms grow at most polynomially in time. Existence of unbounded trajectories only recently [24] received a positive answer in some very special cases, while several instability results were already obtained [9], [22], [23], [26], [21]. Natural model problems for

studying these phenomena seem to be those for which the calculation of solutions is the most explicit, namely integrable systems. Continuing with the example of nonlinear Schrödinger equations, a typical example is the one dimensional cubic nonlinear Schrödinger defocusing equation ([49]). However, in this case, the set of conservation laws is known to control the whole regularity of the solution, so that all the trajectories of $H^s(M)$ are bounded in $H^s(M)$ for every nonnegative integer s. In fact, when the equation is posed on the circle, the recent results of [20] show that, for every such s, the trajectories in $H^s(M)$ are almost periodic in $H^s(M)$.

The goal of this monograph is to study an integrable infinite dimensional system, connected to a nonlinear wave equation, with a dramatically different large time behavior of its trajectories according to the regularity of the phase space.

Following [33] and [48], it is natural to change the dispersion relation by considering the family of equations, $\alpha < 2$,

$$i\partial_t u - |D|^\alpha u = |u|^2 u$$

posed on the circle $\mathbb{S}^1$, where the operator $|D|^\alpha$ is defined by

$$\widehat{|D|^\alpha u}(k) = |k|^\alpha \hat{u}(k)$$

for every distribution $u \in \mathcal{D}'(\mathbb{S}^1)$. Numerical simulations in [33] and [48] suggest weak turbulence. For $1 < \alpha < 2$, it is possible to recover at most polynomial growth of the Sobolev norms ([45]). In the case $\alpha = 1$, the so-called half-wave equation

$$i\partial_t u - |D|u = |u|^2 u \tag{1.0.1}$$

defines a global flow on $H^s(\mathbb{S}^1)$ for every $s \geq \frac{1}{2}$ ([15] — see also [41]). In that case, the only available bound of the H^s-norms is e^{ct^2} ([45]), due to the lack of dispersion. Therefore, the special case $\alpha = 1$ seems to be a more favorable framework for displaying wave turbulence effects.
Notice that this equation can be reformulated as a system, using the Szegő projector Π defined on $\mathcal{D}'(\mathbb{S}^1)$ as

$$\widehat{\Pi u}(k) = \mathbf{1}_{k\geq 0}\hat{u}(k)\ .$$

Indeed, setting $u_+ := \Pi u$, $u_- := (I - \Pi)u$, equation (1.0.1) is equivalent to the system

$$\begin{cases} i(\partial_t + \partial_x)u_+ & = \Pi(|u|^2 u) \\ i(\partial_t - \partial_x)u_- & = (I - \Pi)(|u|^2 u) \end{cases}$$

Furthermore, if the initial datum u_0 satisfies $u_0 = \Pi u_0$ and belongs to $H^s(\mathbb{S}^1)$, $s > 1$, with a small norm ε, then the corresponding solution u is approximated by the solution v of

$$i(\partial_t + \partial_x)v = \Pi(|v|^2 v) \ , \tag{1.0.2}$$

for $|t| \leq \frac{1}{\varepsilon^2} \log \frac{1}{\varepsilon}$ [15] [41]. In other words, the first nonlinear effects in the above system arise through a decoupling of the two equations. Notice that an elementary change of variable in equation (1.0.2) reduces it to

$$i\partial_t u = \Pi(|u|^2 u) \ . \tag{1.0.3}$$

Equation (1.0.3) therefore appears as a model evolution for describing the dynamics of the half-wave equation (1.0.1). We introduced it in [11] under the name *cubic Szegő equation*. In this reference, global wellposedness of the initial value problem was established on

$$H^s_+(\mathbb{S}^1) := H^s(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1) \ , \ s \geq \frac{1}{2} \ ,$$

where $L^2_+(\mathbb{S}^1)$ denotes the range of Π on $L^2(\mathbb{S}^1)$, namely the L^2 functions on the circle with only nonnegative Fourier modes. Notice that $L^2_+(\mathbb{S}^1)$ can be identified to the Hardy space of holomorphic functions on the unit disc $\mathbb{D}$ with L^2 traces on the boundary. Similarly, the phase space $H^{\frac{1}{2}}_+(\mathbb{S}^1)$ can be identified to the Dirichlet space of holomorphic functions on $\mathbb{D}$ with derivative in $L^2(\mathbb{D})$ for the Lebesgue measure. In the sequel, we will freely adopt either the representation on $\mathbb{S}^1$ or the one on $\mathbb{D}$.

The new feature discovered in [11] is that equation (1.0.3) enjoys a Lax pair structure, in connection with Hankel operators — see the end of the introduction for a definition of these operators. As a first consequence, we proved that the H^s norms, for $s > 1$, evolve at most with exponential growth.

Using this Lax pair structure, we also proved that (1.0.3) admits finite dimensional invariant symplectic manifolds of arbitrary dimension, made of rational functions of the variable z in $\mathbb{D}$. Furthermore, the dynamics

on these manifolds is integrable in the sense of Liouville. In [12], we introduced action angle variables on open dense subsets of these invariant manifolds and on a dense G_δ subset of $H^{1/2}_+(\mathbb{S}^1)$. Finally, in [13], we established an explicit formula for all solutions of (1.0.3) which allowed us to prove quasiperiodicity of all trajectories made of rational functions. In particular, this implies that every such trajectory is bounded in all the H^s spaces. However, the explicit formula in [13] was not adapted to study almost periodicity of non rational solutions nor boundedness of non-rational H^s-trajectories for $s > \frac{1}{2}$.

The main purpose of this monograph is to construct a global nonlinear Fourier transform on the space $H^{1/2}_+(\mathbb{S}^1)$ allowing to describe more precisely the cubic Szegő dynamics. As a consequence, we obtain the following results. First we introduce a convenient notation. We denote by Z the nonlinear evolution group defined by (1.0.3) on $H^{1/2}_+(\mathbb{S}^1)$. In other words, for every $u_0 \in H^{1/2}_+(\mathbb{S}^1)$, $t \mapsto Z(t)u_0$ is the solution $u \in C(\mathbb{R}, H^{1/2}_+)$ of equation (1.0.3) such that $u(0) = u_0$. We also set $C^\infty_+(\mathbb{S}^1) := C^\infty(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1)$.

Theorem 1. —

1. *For every $u_0 \in H^{1/2}_+(\mathbb{S}^1)$, the mapping*
$$t \in \mathbb{R} \mapsto Z(t)u_0 \in H^{1/2}_+(\mathbb{S}^1)$$
is almost periodic.

2. *There exists initial data $u_0 \in C^\infty_+(\mathbb{S}^1)$ and sequences $(\bar{t}_n)$, $(\underline{t}^n)$ tending to infinity such that $\forall s > \frac{1}{2}$, $\forall M \in \mathbb{Z}_+$,*
$$\frac{\|Z(\bar{t}_n)u_0\|_{H^s}}{|\bar{t}_n|^M} \underset{n\to\infty}{\longrightarrow} \infty ,$$
and $Z(\underline{t}^n)u_0 \underset{n\to\infty}{\longrightarrow} u_0$ in C^∞_+. Furthermore, the set of such initial data is a dense G_δ subset of $C^\infty_+(\mathbb{S}^1)$.

Remark 1. — – *One could wonder about the influence of the Szegő projector in the equation (1.0.3) and consider the equation without the projector Π*
$$i\partial_t u = |u|^2 u.$$

In that case, $u(t) = \mathrm{e}^{-it|u_0|^2} u_0$ *and*

$$\|u(t)\|_{H^s} \simeq C|t|^s.$$

Hence the action of the Szegő projector both accelerates the energy transfer to high frequencies, and facilitates the transition to low frequencies.

– *The above theorem is an expression of some intermittency phenomenon. Notice that, on the real line, explicit examples with infinite limit at infinity are given in* [40] *with* $\|Z(t)u_0\|_{H^s} \simeq |t|^{2s-1}$.
– *The exponential rate is expected to be optimal but the question remains open for the cubic Szegő equation. However, for the perturbation*

$$i\partial_t u = \Pi(|u|^2 u) + \hat{u}(0),$$

explicit examples such that the Sobolev norms tends exponentially to infinity are given in [46].
– *Existence of unbounded trajectories for the half-wave equation (1.0.1) is an open problem. However, unbounded trajectories have been recently exhibited in* [47] *for*

$$i\partial_t u + \frac{\partial^2}{\partial x^2} u - |D_y| u = |u|^2 u,\ (x, y) \in \mathbb{R}^2$$

by adapting the method developed in [24].

We now give an overview of the nonlinear Fourier transform. First we introduce some additional notation. Given a positive integer n, we set

$$\Omega_n := \{s_1 > s_2 > \cdots > s_n > 0\} \subset \mathbb{R}^n .$$

Given a nonnegative integer $d \geq 0$, we recall that a Blaschke product of degree d is a rational function on $\mathbb{C}$ of the form

$$\Psi(z) = \mathrm{e}^{-i\psi} \prod_{j=1}^{d} \frac{z - p_j}{1 - \overline{p}_j z} \ ,\ \psi \in \mathbb{T}\ ,\ p_j \in \mathbb{D}\ .$$

Alternatively, Ψ can be written as

$$\Psi(z) = \mathrm{e}^{-i\psi} \frac{P(z)}{z^d \overline{P}\left(\frac{1}{z}\right)}\ ,$$

where $\psi \in \mathbb{T}$ is called the angle of Ψ and P is a monic polynomial of degree d with all its roots in $\mathbb{D}$. Such polynomials are called Schur

polynomials. We denote by $\mathcal{B}_d$ the set of Blaschke products of degree d. It is a classical result — see *e.g.* [27] or section 3.5 — that $\mathcal{B}_d$ is diffeomorphic to $\mathbb{T} \times \mathbb{R}^{2d}$.

Given a n-tuple $(d_1, \dots, d_n)$ of nonnegative integers, we set

$$\mathcal{S}_{d_1,\dots,d_n} := \Omega_n \times \prod_{r=1}^{n} \mathcal{B}_{d_r} \ ,$$

endowed with the natural product topology. Given a sequence $(d_r)_{r\geq 1}$ of nonnegative integers, we denote by $\mathcal{S}^{(2)}_{(d_r)}$ the set of pairs $((s_r)_{r\geq 1}, (\Psi_r)_{r\geq 1}) \in \mathbb{R}^\infty \times \prod_{r=1}^{\infty} \mathcal{B}_{d_r}$ such that

$$s_1 > \dots > s_n > \dots > 0 \ , \ \sum_{r=1}^{\infty} (d_r + 1) s_r^2 < \infty \ .$$

We also endow $\mathcal{S}^{(2)}_{(d_r)}$ with the natural topology.

Finally, we denote by $\mathcal{S}_n$ the union of all $\mathcal{S}_{d_1,\dots,d_n}$ over all the n-tuples $(d_1, \dots, d_n)$, and by $\mathcal{S}^{(2)}_\infty$ the union of all $\mathcal{S}^{(2)}_{(d_r)}$ over all the sequences $(d_r)_{r\geq 1}$. Given $(\mathbf{s}, \mathbf{\Psi}) \in \mathcal{S}_n$ and $z \in \mathbb{C}$, we define the matrix $\mathscr{C}(z) := \mathscr{C}(\mathbf{s}, \mathbf{\Psi})(z)$ as follows. If $n = 2q$, the coefficients of $\mathscr{C}(\mathbf{s}, \mathbf{\Psi})(z)$ are given by

$$c_{jk}(z) := \frac{s_{2j-1} - s_{2k} z \Psi_{2k}(z) \Psi_{2j-1}(z)}{s_{2j-1}^2 - s_{2k}^2} \ , \ j, k = 1, \dots, q \ . \tag{1.0.4}$$

If $n = 2q - 1$, we use the same formula as above, with $s_{2q} = 0$.

Theorem 2. — *For every $n \geq 1$, for every $(\mathbf{s}, \mathbf{\Psi}) \in \mathcal{S}_n$, for every $z \in \overline{\mathbb{D}}$, the matrix $\mathscr{C}(\mathbf{s}, \mathbf{\Psi})(z)$ is invertible. We set*

$$u(\mathbf{s}, \mathbf{\Psi})(z) = \langle \mathscr{C}(z)^{-1} (\Psi_{2j-1}(z))_{1\leq j\leq q}, \mathbf{1} \rangle, \tag{1.0.5}$$

where

$$\mathbf{1} := \begin{pmatrix} 1 \\ . \\ . \\ . \\ 1 \end{pmatrix} \ , \ \textit{and } \langle X, Y \rangle := \sum_{k=1}^{q} X_k Y_k.$$

For every $(\mathbf{s}, \mathbf{\Psi}) \in \mathcal{S}_\infty^{(2)}$*, the sequence* $(u_q)_{q\geq 1}$ *with*

$$u_q := u((s_1, \dots, s_{2q}), (\Psi_1, \dots, \Psi_{2q})),$$

is strongly convergent in $H^{1/2}_+(\mathbb{S}^1)$*. We denote its limit by* $u(\mathbf{s}, \mathbf{\Psi})$*.*

The mapping

$$(\mathbf{s}, \mathbf{\Psi}) \in \bigcup_{n=1}^{\infty} \mathcal{S}_n \cup \mathcal{S}_\infty^{(2)} \longmapsto u(\mathbf{s}, \mathbf{\Psi}) \in H^{1/2}_+ \setminus \{0\}$$

is bijective. Furthermore, its restriction to every $\mathcal{S}_{(d_1,\dots,d_n)}$ *and to* $\mathcal{S}_{(d_r)}^{(2)}$ *is a homeomorphism onto its range.*

Finally, the solution at time t *of equation (1.0.3) with initial data* $u_0 = u(\mathbf{s}, \mathbf{\Psi})$ *is* $u(\mathbf{s}, \mathbf{\Psi}(t))$*, where*

$$\Psi_r(t) = \mathrm{e}^{i(-1)^r s_r^2 t}\Psi_r \ .$$

Using Theorem 2, it is easy to prove the first assertion of Theorem 1. As for the second assertion of Theorem 1, it heavily relies on a new phenomenon, which is the loss of continuity of the map $(\mathbf{s}, \mathbf{\Psi}) \mapsto u(\mathbf{s}, \mathbf{\Psi})$ as several consecutive s_r's are collapsing.

Let us explain briefly how the nonlinear Fourier transform is related to spectral analysis. If $u \in H^{1/2}_+(\mathbb{S}^1)$, recall that the Hankel operator of symbol u is the operator $H_u : L^2_+(\mathbb{S}^1) \to L^2_+(\mathbb{S}^1)$ defined by

$$H_u(h) = \Pi(u\overline{h}) \ .$$

It can be shown that H_u^2 is a positive selfadjoint trace class operator. If S is the shift operator defined by

$$Sh(z) = zh(z) \ ,$$

H_u satisfies

$$S^*H_u = H_uS = H_{S^*u} \ .$$

We denote by K_u this new Hankel operator. Let us say that a positive real number s is a singular value associated to u if s^2 is an eigenvalue of H_u^2 or K_u^2. The main point in Theorem 2 is that the list $s_1 > \dots > s_r > \dots$ is the list of singular values associated to $u = u(\mathbf{s}, \mathbf{\Psi})$, and that the corresponding list $\Psi_1, \dots, \Psi_r, \dots$ describes the action of H_u and of K_u on the possibly multidimensional eigenspaces of H_u^2, K_u^2 respectively. This makes more precise a theorem of Adamyan-Arov-Krein about the

structure of Schmidt pairs of Hankel operators [1]. We refer to section 3.5 for more details. As a consequence of Theorem 2, we get inverse spectral theorems on Hankel operators, which generalize to singular values with arbitrary multiplicity the ones we had proved in [12] and [13] for simple singular values. Therefore the nonlinear Fourier transform can be seen as an inverse spectral transform.

We close this introduction by describing the organization of this monograph. In chapter 2 we recall the Lax pair structure discovered in [11] and its application to large time bounds for high Sobolev norms of solutions to the cubic Szegő equation. Chapter 3 is devoted to the study of singular values of Hankel operators H_u and K_u. In particular we introduce the Blaschke products Ψ, which provide the key of the understanding of multiplicity phenomena. Chapter 4 contains the proof of the first part of Theorem 2, establishing the one to one character and the continuity of the nonlinear Fourier transform. Chapter 5 completes the proof of Theorem 2 by describing the Szegő dynamics through the nonlinear Fourier transform, and infers the almost periodicity property of solutions as claimed in the first part of Theorem 1. In Chapter 6, we establish the Baire genericity of unbounded trajectories in H^s for every $s > \frac{1}{2}$ as stated in the second part of Theorem 1. The main ingredient is an instability phenomenon induced by the collapse of consecutive singular values. Finally, Chapter 7 addresses various geometric aspects of our nonlinear Fourier transform in connection to the Szegő hierarchy defined in [11]. Specifically, we show how the nonlinear Fourier transform allows to define action angle variables on special invariant symplectic submanifolds of the phase space, and we discuss the structure of the corresponding Lagrangian tori obtained by freezing the action variables.

Let us mention that part of these results was the content of the preliminary preprint [16], and that another part was announced in the proceedings paper [17].

CHAPTER 2

HANKEL OPERATORS AND THE LAX PAIR STRUCTURE

As we pointed out in the introduction, Hankel operators arise naturally to understand the cubic Szegő equation. In this chapter, we first give some basic definition and properties of Hankel operators and we recall the Lax pair structure of the cubic Szegő equation. Finally, we show how to use this structure to get large time bounds for high Sobolev norms of solutions to the cubic Szegő equation.

2.1. Hankel operators H_u and K_u

Let $u \in H^{\frac{1}{2}}_+(\mathbb{S}^1)$. We denote by H_u the $\mathbb{C}$–antilinear operator defined on $L^2_+(\mathbb{S}^1)$ as

$$H_u(h) = \Pi(u\overline{h})\ ,\ h \in L^2_+(\mathbb{S}^1)\ .$$

In terms of Fourier coefficients, this operator reads

$$\widehat{H_u(h)}(n) = \sum_{p=0}^{\infty} \hat{u}(n+p)\overline{\hat{h}(p)}\ .$$

In particular, its Hilbert–Schmidt norm

$$\|H_u\|_{HS} = (\sum_{n,p\geq 0} |\hat{u}(n+p)|^2)^{1/2} = (\sum_{\ell\geq 0}(1+\ell)|\hat{u}(\ell)|^2)^{1/2} \simeq \|u\|_{H^{1/2}}$$

is finite for $u \in H^{\frac{1}{2}}_+(\mathbb{S}^1)$ and

$$\|H_u\|_{\mathcal{L}(L^2_+)} \leq \|H_u\|_{HS}\ . \tag{2.1.1}$$

We call H_u the Hankel operator of symbol u. It is well known from Kronecker's theorem, [30], [37], [39], that H_u is of finite rank if and only if u is a rational function without poles in the closure of the unit disc.

In fact, the definition of Hankel operators may be extended to a larger class of symbol. By the Nehari theorem ([36]), H_u is well defined and bounded on $L^2_+(\mathbb{S}^1)$ if and only if u belongs to $\Pi(L^\infty(\mathbb{S}^1))$ or equivalently to $BMO_+(\mathbb{S}^1)$. Moreover, by the Hartman theorem ([25]), it is a compact operator if and only if u is the projection of a continuous function on the torus, or equivalently if and only if it belongs to $VMO_+(\mathbb{S}^1)$ with equivalent norms. In the following, we will mainly consider the class of Hilbert-Schmidt Hankel operators. We will generalize part of our result to compact Hankel operators in section 4.3.

Notice that this definition of Hankel operators is different from the standard ones used in references [37], [39], where Hankel operators are rather defined as linear operators from L^2_+ into its orthogonal complement. The link between these two definitions can be easily established by means of the involution

$$f^\sharp(\mathrm{e}^{ix}) = \mathrm{e}^{-ix}\overline{f(\mathrm{e}^{ix})}\ .$$

Notice also that H_u satisfies the following self adjointness identity,

$$(H_u(h_1)|h_2) = (H_u(h_2)|h_1)\ ,\ \ h_1, h_2 \in L^2_+(\mathbb{S}^1)\ . \tag{2.1.2}$$

In particular, H_u is a $\mathbb{R}$-linear symmetric operator for the real inner product

$$\langle h_1, h_2\rangle := \mathrm{Re}(h_1|h_2)\ .$$

A fundamental property of Hankel operators is their connection with the shift operator S, defined on $L^2_+(\mathbb{S}^1)$ as

$$Su(\mathrm{e}^{ix}) = \mathrm{e}^{ix}u(\mathrm{e}^{ix})\ .$$

This property reads

$$S^*H_u = H_uS = H_{S^*u}\ ,$$

where S^* denotes the adjoint of S. We denote by K_u this operator, and call it the shifted Hankel operator of symbol u. Hence

$$K_u := S^*H_u = H_uS = H_{S^*u}\ . \tag{2.1.3}$$

Notice that, for $u \in H^{1/2}_+(\mathbb{S}^1)$, K_u is Hilbert–Schmidt and symmetric as well. As a consequence, operators H_u^2 and K_u^2 are $\mathbb{C}$–linear trace class positive operators on $L^2_+(\mathbb{S}^1)$. Recall that the singular values of H_u and K_u correspond to the square roots of the eigenvalues of the self-adjoint positive operators H_u^2 and K_u^2. Moreover, operators H_u^2 and K_u^2 are related by the following important identity,

$$K_u^2 = H_u^2 - (\cdot|u)u \ . \tag{2.1.4}$$

Let us consider the special case where H_u is an operator of finite rank N. From identity (2.1.2), H_u and H_u^2 have the same kernel, hence have the same rank. Then (2.1.4) implies that the rank of K_u is $N-1$ or N.

Definition 1. — *If d is a positive integer, we denote by $\mathcal{V}(d)$ the set of symbols u such that the sum of the rank of H_u and of the rank of K_u is d.*

The Kronecker theorem can be made more precise by the following statement, see the appendix of [11]. The set $\mathcal{V}(d)$ is a complex Kähler d–dimensional submanifold of $L^2_+(\mathbb{S}^1)$, which consists of functions of the form

$$u(\mathrm{e}^{ix}) = \frac{A(\mathrm{e}^{ix})}{B(\mathrm{e}^{ix})} \ ,$$

where A, B are polynomials with no common factors, B has no zero in the closed unit disc, $B(0) = 1$, and

- If $d = 2N$ is even, the degree of A is at most $N-1$ and the degree of B is exactly N.
- If $d = 2N-1$ is odd, the degree of A is exactly $N-1$ and the degree of B is at most $N-1$.

2.2. The Lax pair structure

In this section, we recall the Lax pairs associated to the cubic Szegő equation, see [11], [12]. First we introduce the notion of a Toeplitz operator. Given $b \in L^\infty(\mathbb{S}^1)$, we define $T_b : L^2_+ \to L^2_+$ as

$$T_b(h) = \Pi(bh) \ , \ h \in L^2_+ \ . \tag{2.2.1}$$

Notice that T_b is bounded and $T_b^* = T_{\bar b}$. The starting point is the following lemma.

Lemma 1. — *Let $a, b, c \in H^s_+$, $s > \frac{1}{2}$. Then*

$$H_{\Pi(a\bar b c)} = T_{a\bar b} H_c + H_a T_{b\bar c} - H_a H_b H_c \ .$$

Proof. — Given $h \in L^2_+$, we have

$$\begin{aligned} H_{\Pi(a\bar b c)}(h) &= \Pi(a\bar b c\bar h) = \Pi(a\bar b \Pi(c\bar h)) + \Pi(a\bar b (I-\Pi)(c\bar h)) \\ &= T_{a\bar b} H_c(h) + H_a(g) \ , \ g := b\overline{(I-\Pi)(c\bar h)} \ . \end{aligned}$$

Since $g \in L^2_+$,

$$g = \Pi(g) = \Pi(b\bar c h) - \Pi(b\overline{\Pi(c\bar h)}) = T_{b\bar c}(h) - H_b H_c(h) \ .$$

This completes the proof. □

Using Lemma 1 with $a = b = c = u$, we get

$$H_{\Pi(|u|^2 u)} = T_{|u|^2} H_u + H_u T_{|u|^2} - H_u^3 \ . \tag{2.2.2}$$

Theorem 3. — *Let $u \in C^\infty(\mathbb{R}, H^s_+), s > \frac{1}{2}$, be a solution of (1.0.3). Then*

$$\begin{aligned} \frac{dH_u}{dt} &= [B_u, H_u] \ , \ B_u := \frac{i}{2} H_u^2 - iT_{|u|^2} \ , \\ \frac{dK_u}{dt} &= [C_u, K_u] \ , \ C_u := \frac{i}{2} K_u^2 - iT_{|u|^2} \ . \end{aligned}$$

Proof. — Using equation (1.0.3) and identity (2.2.2),

$$\frac{dH_u}{dt} = H_{-i\Pi(|u|^2 u)} = -iH_{\Pi(|u|^2 u)} = -i(T_{|u|^2} H_u + H_u T_{|u|^2} - H_u^3) \ .$$

Using the antilinearity of H_u, this leads to the first identity. For the second one, we observe that

$$K_{\Pi(|u|^2 u)} = H_{\Pi(|u|^2 u)} S = T_{|u|^2} H_u S + H_u T_{|u|^2} S - H_u^3 S \ . \tag{2.2.3}$$

Moreover, notice that

$$T_b(Sh) = ST_b(h) + (bSh|1) \ .$$

In the case $b = |u|^2$, this gives

$$T_{|u|^2} Sh = ST_{|u|^2} h + (|u|^2 Sh|1) \ .$$

Moreover,

$$(|u|^2 Sh|1) = (u|u\overline{Sh}) = (u|K_u(h)) \ .$$

Consequently,

$$H_u T_{|u|^2} Sh = K_u T_{|u|^2} h + (K_u(h)|u)u \ .$$

Coming back to (2.2.3), we obtain

$$K_{\Pi(|u|^2 u)} = T_{|u|^2} K_u + K_u T_{|u|^2} - (H_u^2 - (\cdot|u)u) K_u \ .$$

Using identity (2.1.4), this leads to

$$K_{\Pi(|u|^2 u)} = T_{|u|^2} K_u + K_u T_{|u|^2} - K_u^3 \ . \tag{2.2.4}$$

The second identity is therefore a consequence of antilinearity and of

$$\frac{dK_u}{dt} = -iK_{\Pi(|u|^2 u)} \ .$$

□

In the sequel, we denote by $\mathcal{L}(L^2_+)$ the Banach space of bounded linear operators on L^2_+. Observing that B_u, C_u are linear and antiselfadjoint, we obtain, following a classical argument due to Lax [31],

Corollary 1. — *Under the conditions of Theorem 3, define $U = U(t)$, $V = V(t)$ the solutions of the following linear ODEs on $\mathcal{L}(L^2_+)$,*

$$\frac{dU}{dt} = B_u U \ , \ \frac{dV}{dt} = C_u V \ , \ U(0) = V(0) = I \ .$$

Then $U(t), V(t)$ are unitary operators and

$$H_{u(t)} = U(t) H_{u(0)} U(t)^* \ , \ K_{u(t)} = V(t) K_{u(0)} V(t)^* \ .$$

In other words, the Hankel operators H_u and K_u remain unitarily equivalent to the Hankel operators associated to their initial data under the cubic Szegő flow for initial datum in $H^s(\mathbb{S}^1)$, $s > 1/2$. In particular, the cubic Szegő flow preserves the eigenvalues of H_u^2 and of K_u^2, hence the singular values of H_u and K_u. By a standard continuity argument, this result extends to initial data in $H^{1/2}_+(\mathbb{S}^1)$. We recover that the Hilbert-Schmidt norm– being the ℓ^2-norm of the singular values of H_u– hence the $H^{1/2}_+$ norm of the symbol is a conservation law. It is therefore natural to study the spectral properties of these Hankel operators.

As a special case, the ranks of H_u and K_u are conserved by the cubic Szegő flow, which therefore acts as a Hamiltonian flow on all the symplectic manifolds $\mathcal{V}(d)$.

2.3. Application: an exponential bound for Sobolev norms

In this short section, we show how the Lax pair structure, combined with harmonic analysis results on Hankel operators, lead to an a priori bound on the possible long time growth of the Sobolev norms of the solution.

From the work of Peller ([39]), a Hankel operator belongs to the Schatten class S_p, $1 \leq p < \infty$, if and only if its symbol belongs to the Besov space $B^{1/p}_{p,p}(\mathbb{S}^1)$, with

$$\mathrm{Tr}(|H_u|^p) \simeq \|u\|^p_{B^{1/p}_{p,p}} \ .$$

In particular, since, for every $s > 1$, $H^s_+(\mathbb{S}^1) \subset B^1_{1,1}(\mathbb{S}^1) \subset L^\infty(\mathbb{S}^1)$, we obtain

$$\|u\|_{L^\infty} \leq A\mathrm{Tr}(|H_u|) \leq B_s\|u\|_{H^s}, \ s > 1. \tag{2.3.1}$$

This provides the following result, already observed in [11].

Theorem 4. — *([11]). For every datum $u_0 \in H^s_+$ for some $s > 1$, the solution is bounded in L^∞, with*

$$\sup_{t\in\mathbb{R}} \|Z(t)u_0\|_{L^\infty} \leq C_s\|u_0\|_{H^s} \ . \tag{2.3.2}$$

Furthermore,

$$\|Z(t)u_0\|_{H^s} \leq \|C'_s u_0\|_{H^s}\mathrm{e}^{C'_s\|u_0\|^2_{H^s}|t|}, \ s > 1. \tag{2.3.3}$$

Proof. — The first statement is an immediate consequence of (2.3.1) and of the conservation law

$$\mathrm{Tr}(|H_{Z(t)u_0}|) = \mathrm{Tr}(|H_{u_0}|) \ .$$

From Duhamel formula, $u(t) := Z(t)u_0$ is given by

$$u(t) := u_0 - i\int_0^t \Pi(|u|^2u)(t')dt' \ ,$$

so that

$$\begin{aligned}
\|u(t)\|_{H^s} &\leq \|u_0\|_{H^s} + \int_0^t \||u(t')|^2 u(t')\|_{H^s} dt' \\
&\leq \|u_0\|_{H^s} + D_s \int_0^t \|u(t')\|_{L^\infty}^2 \|u(t')\|_{H^s} dt \\
&\leq \|u_0\|_{H^s} + D_s C'_s \int_0^t \|u_0\|_{H^s}^2 \|u(t')\|_{H^s} dt
\end{aligned}$$

and the bound (2.3.3) follows from the Gronwall lemma. □

CHAPTER 3

SPECTRAL ANALYSIS

In this chapter, we establish several properties of singular values of Hankel operators H_u and K_u introduced in Chapter 1. In particular, we prove – see Lemma 2 — that, for a given singular value of either H_u or K_u, the difference of the multiplicities is 1. In section 3.2 we review trace and determinant formulae relating the singular values to other norming constants. In section 3.4, we revisit two important theorems by Adamyan–Arov–Krein on the structure of Schmidt pairs for Hankel operators, and on approximation of arbitrary symbols by rational symbols. Section 3.5 is the most important one of this chapter. Choosing special Schmidt pairs, we construct Blaschke products which allow to describe the action of H_u and K_u on the eigenspaces of H_u^2 and K_u^2.

3.1. Spectral decomposition of the operators H_u and K_u

This section is devoted to a precise spectral analysis of operators H_u^2 and K_u^2 on the closed range of H_u. This spectral analysis is closely related to the construction of our non linear Fourier transform.

For every $s \geq 0$ and $u \in VMO_+(\mathbb{S}^1) := VMO(\mathbb{S}^1) \cap L^2_+(\mathbb{S}^1)$, we set

$$E_u(s) := \ker(H_u^2 - s^2 I)\ ,\ F_u(s) := \ker(K_u^2 - s^2 I)\ . \tag{3.1.1}$$

Notice that $E_u(0) = \ker H_u\ ,\ F_u(0) = \ker K_u$. Moreover, from the compactness of H_u, if $s > 0$, $E_u(s)$ and $F_u(s)$ are finite dimensional. Using (2.1.3) and (2.1.4), one can show the following result.

Lemma 2. — *Let $u \in VMO_+(\mathbb{S}^1) \setminus \{0\}$ and $s > 0$ such that*

$$E_u(s) + F_u(s) \neq \{0\} \ .$$

Then one of the following properties holds.

1. $\dim E_u(s) = \dim F_u(s) + 1$, *$u \not\perp E_u(s)$, and $F_u(s) = E_u(s) \cap u^\perp$.*
2. $\dim F_u(s) = \dim E_u(s) + 1$, *$u \not\perp F_u(s)$, and $E_u(s) = F_u(s) \cap u^\perp$.*

Proof. — Let $s > 0$ be such that $E_u(s) + F_u(s) \neq \{0\}$. We first claim that either $u \perp E_u(s)$ or $u \perp F_u(s)$. Assume first $u \not\perp E_u(s)$, then there exists $h \in E_u(s)$ such that $(h|u) \neq 0$. From equation (2.1.4),

$$-(h|u)u = (K_u^2 - s^2 I)h \in (F_u(s))^\perp \ ,$$

hence $u \perp F_u(s)$. Similarly, if $u \not\perp F_u(s)$, then $u \perp E_u(s)$.
Assume $u \perp F_u(s)$. Then, for any $h \in F_u(s)$, as $K_u^2 = H_u^2 - (\cdot|u)u$, $H_u^2(h) = K_u^2(h) = s^2 h$, hence $F_u(s) \subset E_u(s)$. We claim that this inclusion is strict. Indeed, suppose it is an equality. Then H_u and K_u are both automorphisms of the vector space

$$N := F_u(s) = E_u(s) \ .$$

Consequently, since $K_u = S^* H_u$, $S^*(N) \subset N$. On the other hand, since every $h \in N$ is orthogonal to u, we have

$$0 = (H_u(h)|u) = (1|H_u^2 h) = s^2 (1|h) \ ,$$

hence $N \perp 1$. Therefore, for every $h \in N$, for every integer k, $(S^*)^k(h) \perp 1$. Since $S^k(1) = z^k$, we conclude that all the Fourier coefficients of h are 0, hence $N = \{0\}$, a contradiction. Hence, the inclusion of $F_u(s)$ in $E_u(s)$ is strict and, necessarily $u \not\perp E_u(s)$ and $F_u(s) = E_u(s) \cap u^\perp$. One also has $\dim E_u(s) = \dim F_u(s) + 1$.
One proves as well that if $u \perp E_u(s)$ then $u \not\perp F_u(s)$, $E_u(s) = F_u(s) \cap u^\perp$ and $\dim F_u(s) = \dim E_u(s) + 1$. This gives the result. □

We define

$$\Sigma_H(u) := \{s \geq 0;\ u \not\perp E_u(s)\}\,, \tag{3.1.2}$$

$$\Sigma_K(u) := \{s \geq 0;\ u \not\perp F_u(s)\}\,. \tag{3.1.3}$$

Remark first that $0 \notin \Sigma_H(u)$, since $u = H_u(1)$ belongs to the range of H_u hence, is orthogonal to its kernel. From Lemma 2, if $s \in \Sigma_H(u)$

then $\dim E_u(s) = \dim F_u(s) + 1$ and if $s \in \Sigma_K(u) \setminus \{0\}$, $\dim F_u(s) = \dim E_u(s) + 1$. Consequently, $\Sigma_H(u)$ and $\Sigma_K(u)$ are disjoint. We will use the following terminology.
If $s \in \Sigma_H(u)$, we say that s is *H-dominant*.
If $s \in \Sigma_K(u) \setminus \{0\}$, we say that s is *K-dominant*.
Elements of the set $\Sigma_H(u) \cup (\Sigma_K(u) \setminus \{0\})$ are called *singular values associated to u*. If s is such a singular value, the *dominant multiplicity* of s is defined to be the maximum of $\dim E_u(s)$ and of $\dim F_u(s)$.

Lemma 3. — 1. *$\Sigma_H(u)$ and $\Sigma_K(u)$ have the same cardinality.*

2. *If (ρ_j) denotes the decreasing sequence of elements of $\Sigma_H(u)$, and (σ_k) the decreasing sequence of elements of $\Sigma_K(u)$, then*

$$\rho_1 > \sigma_1 > \rho_2 > \dots$$

Proof. — Since $K_u^2 = H_u^2 - (\,.\,|u)u$, the cyclic spaces generated by u under the action of H_u^2 and K_u^2, namely

$$\langle u\rangle_{H_u^2} := \text{clos span}\{H_u^{2k},\ k \in \mathbb{N}\}$$

$$\langle u\rangle_{K_u^2} := \text{clos span}\{K_u^{2k},\ k \in \mathbb{N}\}$$

are equal. By the spectral theory of H_u^2 and of K_u^2, we have the orthogonal decompositions,

$$L^2_+ = \overline{\oplus_{s\geq 0} E_u(s)} = \overline{\oplus_{s\geq 0} F_u(s)}\ .$$

Writing u according to these two orthogonal decompositions yields

$$u = \sum_{\rho\in\Sigma_H(u)} u_\rho = \sum_{\sigma\in\Sigma_K(u)} u'_\sigma\ .$$

Consequently, the cyclic spaces are given by

$$\langle u\rangle_{H_u^2} = \overline{\oplus_{\rho\in\Sigma_H(u)}\mathbb{C}u_\rho}\ ,\ \langle u\rangle_{K_u^2} = \overline{\oplus_{\sigma\in\Sigma_K(u)}\mathbb{C}u'_\sigma}\ .$$

This proves that $\Sigma_H(u)$ and $\Sigma_K(u)$ have the same — possibly infinite — number of elements.

Now, we show that the singular values are alternatively H-dominant and K-dominant. Recall that the singular values of a bounded operator T on a Hilbert space $\mathcal{H}$, are given by the following min-max formula. For

every $m \geq 1$, denote by $\mathcal{F}_m$ the set of linear subspaces of $\mathcal{H}$ of dimension at most m. The m-th singular value of T is given by

$$\lambda_m(T) = \min_{F \in \mathcal{F}_{m-1}} \max_{f \in F^\perp, \|f\|=1} \|T(f)\|. \tag{3.1.4}$$

Using equation (2.1.4) and this formula, we get

$$\lambda_1(H_u^2) \geq \lambda_1(K_u^2) \geq \lambda_2(H_u^2) \geq \lambda_2(K_u^2) \geq \dots$$

Let $s_1^2 = \lambda_1(H_u^2)$. We claim that s_1 is H-dominant. Indeed, denote by m_1 the dimension of $E_u(s_1)$. Hence,

$$s_1^2 = \lambda_1(H_u^2) = \dots = \lambda_{m_1}(H_u^2) > \lambda_{m_1+1}(H_u^2).$$

Since

$$\lambda_{m_1+1}(K_u^2) \leq \lambda_{m_1+1}(H_u^2) < s_1^2,$$

the dimension of $F_u(s_1)$ is at most m_1. Hence by Lemma 2, it is exactly $m_1 - 1$. Let $s_2^2 = \lambda_{m_1}(K_u^2)$. Denote by m_2 the dimension of $F_u(s_2)$ so that

$$s_2^2 = \lambda_{m_1}(K_u^2) = \dots = \lambda_{m_1+m_2-1}(K_u^2) > \lambda_{m_1+m_2}(K_u^2) \geq \lambda_{m_1+m_2+1}(H_u^2).$$

As before, it implies that the dimension of $E_u(s_2)$ is at most m_2. By Lemma 2, the dimension of $E_u(s_2)$ is $m_2 - 1$ and s_2 is K-dominant. An easy induction argument allows to conclude.

□

At this stage, we introduce special classes of functions on the circle, connected to the properties of the associated singular values. Given a finite sequence $(d_1, \dots, d_n)$ of nonnegative integers, we denote by $\mathcal{V}_{(d_1,\dots,d_n)}$ the class of functions u having n associated singular values $s_1 > \dots > s_n$, of dominant multiplicities $d_1 + 1, \dots, d_n + 1$ respectively. In view of definition 1, we observe that

$$\mathcal{V}_{(d_1,\dots,d_n)} \subset \mathcal{V}(d) \ , \ d = n + 2\sum_{r=1}^{n} d_r \ . \tag{3.1.5}$$

Similarly, given a sequence $(d_r)_{r\geq 1}$ of nonnegative integers, $\mathcal{V}^{(2)}_{(d_r)_{r\geq 1}}$ the class of functions $u \in H^{\frac{1}{2}}(\mathbb{S}^1)$ having a decreasing sequence of associated singular values with dominant multiplicities $(d_r + 1)_{r\geq 1}$.

Lemma 4. — *Let $\rho \in \Sigma_H(u)$ and $\sigma \in \Sigma_K(u)$; let u_ρ and u'_σ denote respectively the orthogonal projections of u onto $E_u(\rho)$, and onto $F_u(\sigma)$, then*

1. *If $\rho \in \Sigma_H(u)$,*

$$u_\rho = \|u_\rho\|^2 \sum_{\sigma\in\Sigma_K(u)} \frac{u'_\sigma}{\rho^2-\sigma^2} \ , \tag{3.1.6}$$

2. *If $\sigma \in \Sigma_K(u)$,*

$$u'_\sigma = \|u'_\sigma\|^2 \sum_{\rho\in\Sigma_H(u)} \frac{u_\rho}{\rho^2-\sigma^2} \ . \tag{3.1.7}$$

3. *A nonnegative number σ belongs to $\Sigma_K(u)$ if and only if it does not belong to $\Sigma_H(u)$ and*

$$\sum_{\rho\in\Sigma_H(u)} \frac{\|u_\rho\|^2}{\rho^2-\sigma^2} = 1 \ . \tag{3.1.8}$$

Proof. — Let us prove (3.1.6) and (3.1.7). Observe that, by the Fredholm alternative, for $\sigma > 0$, $H_u^2 - \sigma^2 I$ is an automorphism of $E_u(\sigma)^\perp$. Consequently, if moreover $\sigma \in \Sigma_K(u)$, $u \in E_u(\sigma)^\perp$ and there exists $v \in E_u(\sigma)^\perp$ unique such that

$$(H_u^2 - \sigma^2 I)v = u \ .$$

We set $v := (H_u^2 - \sigma^2 I)^{-1}(u)$. If $\sigma = 0 \in \sigma_K(u)$, of course H_u^2 is no more a Fredholm operator. However let us prove that there still exists $w \in E_u(0)^\perp$ such that

$$H_u^2(w) = u \ .$$

Indeed, $E_u(0) = \ker H_u^2 = \ker H_u$, $F_u(0) = \ker K_u^2 = \ker K_u$, and since $K_u = S^*H_u$, therefore $E_u(0) \subset F_u(0)$. As $u = H_u(1) \in \mathrm{Ran}(H_u) \subset E_u(0)^\perp$, the hypothesis $u \not\perp F_u(0)$ implies that the inclusion $E_u(0) \subset F_u(0)$ is strict. This means that there exists $w \in F_u(0) \cap E_u(0)^\perp$, $w \neq 0$. It means $S^*H_u(w) = 0$ hence $H_u w$ is a constant function which is not zero since $w \in E_u(0)^\perp$. Multiplying w by a convenient complex number we may assume that $H_u(w) = 1$, whence $H_u^2(w) = u$, and this characterizes $w \in E_u(0)^\perp$. Again we set $w := (H_u^2)^{-1}(u)$.

For every $\sigma \in \Sigma_K(u)$, the equation

$$K_u^2 h = \sigma^2 h$$

is equivalent to

$$(H_u^2 - \sigma^2 I)h = (h|u)u \ ,$$

or $h \in \mathbb{C}(H_u^2 - \sigma^2 I)^{-1}(u) \oplus E_u(\sigma)$, with

$$((H_u^2 - \sigma^2 I)^{-1}(u)|u) = 1 \ . \tag{3.1.9}$$

We apply this property to $h = u'_\sigma$. Since $u'_\sigma \in E_u(\sigma)^\perp$, there exists $\lambda \in \mathbb{C}$ with $u'_\sigma = \lambda(H_u^2 - \sigma^2 I)^{-1}(u)$. Applying equation (3.1.9), this leads to

$$\frac{u'_\sigma}{\|u'_\sigma\|^2} = (H_u^2 - \sigma^2 I)^{-1}(u) \ .$$

In particular, if $\rho \in \Sigma_H(u)$, $\sigma \in \Sigma_K(u)$,

$$\left(\frac{u'_\sigma}{\|u'_\sigma\|^2}\middle|\frac{u_\rho}{\|u_\rho\|^2}\right) = \frac{1}{\rho^2 - \sigma^2} \ .$$

This leads to equations (3.1.6) and (3.1.7). Finally, equation (3.1.8) is nothing but the expression of (3.1.9) in view of equation (3.1.6).

This completes the proof of Lemma 4. $\square$

3.2. Some Bateman-type formulae

In this section, we establish some formulae linking the singular values of a Hankel operator and its shifted operator.
Let $u \in VMO_+(\mathbb{S}^1)$. For $\rho \in \Sigma_H(u)$, we denote by $\sigma(\rho)$ the biggest element of $\Sigma_K(u)$ which is smaller than ρ. When $\displaystyle\prod_{\rho \in \Sigma_H(u)} \frac{\sigma(\rho)^2}{\rho^2} = 0$, we define $\rho(\sigma)$ to be the biggest element of $\Sigma_H(u)$ which is smaller than σ. This is always well defined for infinite sequences tending to zero. In the case of finite sequence, the hypothesis implies that $0 \in \Sigma_K(u)$ and lemma 2 and lemma 3 allows to define $\rho(\sigma)$ in that case for any $\sigma \in \Sigma_k(u) \setminus \{0\}$.
In this section, we state and prove several formulae connecting these sequences. These formulae are based on the special case of a general formula for the resolvent of a finite rank perturbation of an operator, which seems to be due to Bateman [4] in the framework of Fredholm integral equations. Further references can be found in Chap. II, sect. 4.6 of [28], section 106 of [2] and [35], from which we borrowed this information.

Proposition 1. — *The following functions coincide respectively for x outside the set $\{\frac{1}{\rho^2}\}_{\rho\in\Sigma_H(u)}$ and outside the set $\{\frac{1}{\sigma^2}\}_{\sigma\in\Sigma_K(u)\setminus\{0\}}$.*

$$\prod_{\rho\in\Sigma_H(u)} \frac{1-x\sigma(\rho)^2}{1-x\rho^2} = 1 + x \sum_{\rho\in\Sigma_H(u)} \frac{\|u_\rho\|^2}{1-x\rho^2} \tag{3.2.1}$$

$$\prod_{\rho\in\Sigma_H(u)} \frac{1-x\rho^2}{1-x\sigma(\rho)^2} = 1 - x \left(\sum_{\sigma\in\Sigma_K(u)} \frac{\|u'_\sigma\|^2}{1-x\sigma^2}\right) . \tag{3.2.2}$$

Furthermore,

$$1 - \sum_{\rho\in\Sigma_H(u)} \frac{\|u_\rho\|^2}{\rho^2} = \prod_{\rho\in\Sigma_H(u)} \frac{\sigma(\rho)^2}{\rho^2} , \tag{3.2.3}$$

and, if $\displaystyle\prod_{\rho\in\Sigma_H(u)} \frac{\sigma(\rho)^2}{\rho^2} = 0$,

$$\sum_{\rho\in\Sigma_H(u)} \frac{\|u_\rho\|^2}{\rho^4} = \frac{1}{\rho_{\max}^2} \prod_{\sigma\in\Sigma_K(u)\setminus\{0\}} \frac{\sigma^2}{\rho(\sigma)^2} . \tag{3.2.4}$$

Here $\rho_{\max}$ denotes the biggest element of $\Sigma_H(u)$. For $\rho \in \Sigma_H(u)$, the $\|u_\rho\|^2$'s are given by

$$\|u_\rho\|^2 = \left(\rho^2 - \sigma(\rho)^2\right) \prod_{\rho'\neq\rho} \frac{\rho^2-\sigma(\rho')^2}{\rho^2-\rho'^2} . \tag{3.2.5}$$

and the $\|u'_{\sigma(\rho)}\|^2$'s are given by

$$\|u'_{\sigma(\rho)}\|^2 = (\rho^2 - \sigma(\rho)^2) \prod_{\rho'\neq\rho} \frac{\sigma(\rho)^2-\rho'^2}{\sigma(\rho)^2-\sigma(\rho')^2}. \tag{3.2.6}$$

The kernel of K_u is non trivial if and only if $\displaystyle\prod_{\rho\in\Sigma_H(u)} \frac{\sigma(\rho)^2}{\rho^2} = 0$, *and in that case*

$$\|u'_0\|^2 = \rho_{\max}^2 \prod_{\sigma\in\Sigma_K(u)\setminus\{0\}} \frac{\rho(\sigma)^2}{\sigma^2}$$

Proof. — For $x \notin \{\frac{1}{\rho^2}\}_{\rho\in\Sigma_H(u)}$, we set

$$J(x) := ((I - xH_u^2)^{-1}(1)|1).$$

We claim that

$$J(x) = \prod_{\rho \in \Sigma_H(u)} \frac{1 - x\sigma(\rho)^2}{1 - x\rho^2}. \tag{3.2.7}$$

Indeed, let us first assume that H_u^2 and K_u^2 are of trace class and compute the trace of $(I - xH_u^2)^{-1} - (I - xK_u^2)^{-1}$. We write

$$[(I - xH_u^2)^{-1} - (I - xK_u^2)^{-1}](f) = \frac{x}{J(x)}(f|(I - xH_u^2)^{-1}u) \cdot (I - xH_u^2)^{-1}u.$$

Consequently, taking the trace, we get

$$\mathrm{Tr}[(I - xH_u^2)^{-1} - (I - xK_u^2)^{-1}] = \frac{x}{J(x)} \|(I - xH_u^2)^{-1}u\|^2.$$

Since, on the one hand

$$\|(I - xH_u^2)^{-1}u\|^2 = ((I - xH_u^2)^{-1}H_u^2(1)|1) = J'(x)$$

and on the other hand

$$\begin{aligned} \mathrm{Tr}[(I - xH_u^2)^{-1} - (I - xK_u^2)^{-1}] &= x\mathrm{Tr}[H_u^2(I - xH_u^2)^{-1} - K_u^2(I - xK_u^2)^{-1}] \\ &= x \sum_{\rho \in \Sigma_H(u)} \left(\frac{\rho^2}{1 - \rho^2 x} - \frac{\sigma(\rho)^2}{1 - \sigma(\rho)^2 x} \right), \end{aligned}$$

where we used lemmas 3 and 4. On the other hand,

$$\sum_{\rho \in \Sigma_H(u)} \left(\frac{\rho^2}{1 - \rho^2 x} - \frac{\sigma(\rho)^2}{1 - \sigma(\rho)^2 x} \right) = \frac{J'(x)}{J(x)}, \; x \notin \left\{ \frac{1}{\rho^2}, \frac{1}{\sigma(\rho)^2} \right\}_{\rho \in \Sigma_H(u)}. \tag{3.2.8}$$

This gives equality (3.2.7) for H_u^2 and K_u^2 of trace class. To extend this formula to compact operators, we remark that $\sum_{\rho \in \Sigma_H(u)} (\rho^2 - \sigma(\rho)^2)$ is summable since, in case of infinite sequences, we have $0 \le \rho^2 - \sigma(\rho)^2 \le \rho^2 - \rho(\sigma(\rho))^2$ so that it leads to a telescopic serie which converges since (ρ^2) tends to zero from the compactness of H_u^2. Hence the infinite product in formula (3.2.7) and the above computation makes sense for compact operators.

On the other hand, for $x \notin \{\frac{1}{\rho^2}\}$,

$$\begin{aligned} J(x) &= ((I - xH_u^2)^{-1}(1)|1) = 1 + x((I - xH_u^2)^{-1}(u)|u) \\ &= 1 + x(\sum_\rho (I - xH_u^2)^{-1}(u_\rho)|u)) = 1 + x \sum_{\rho \in \Sigma_H(u)} \frac{\|u_\rho\|^2}{1 - x\rho^2} \end{aligned}$$

hence

$$\prod_{\rho\in\Sigma_H(u)}\frac{1-x\sigma(\rho)^2}{1-x\rho^2}=1+x\sum_{\rho\in\Sigma_H(u)}\frac{\|u_\rho\|^2}{1-x\rho^2}\ . \tag{3.2.9}$$

Passing to the limit as x goes to $-\infty$ in (3.2.1), we obtain (3.2.3). If we assume that the left hand side of (3.2.3) cancels, then (3.2.1) can be rewritten as

$$\prod_{\rho\in\Sigma_H(u)}\frac{1-x\sigma(\rho)^2}{1-x\rho^2}=\sum_{\rho\in\Sigma_H(u)}\frac{\|u_\rho\|^2}{\rho^2(1-x\rho^2)}.$$

Multiplying by x and passing to the limit as x goes to $-\infty$ in this new identity, we obtain (3.2.4). Multiplying both terms of (3.2.1) by $(1-x\rho^2)$ and letting x go to $1/\rho^2$. We get

$$\|u_\rho\|^2=\left(\rho^2-\sigma(\rho)^2\right)\prod_{\rho'\neq\rho}\frac{\rho^2-\sigma(\rho')^2}{\rho^2-\rho'^2}.$$

For Equality (4.1.13), we do almost the same analysis. First, we establish as above that

$$\frac{1}{J(x)}=1-x((I-xK_u^2)^{-1}(u)|u)=1-x\sum_{\sigma\in\Sigma_K(u)}\frac{\|u'_\sigma\|^2}{1-x\sigma^2}.$$

Identifying this expression with

$$\frac{1}{J(x)}=\prod_\rho\frac{1-x\rho^2}{1-x\sigma(\rho)^2}$$

we get, for $\rho\in\Sigma_H(u)$

$$\|u'_{\sigma(\rho)}(\rho)\|^2=(\rho^2-\sigma(\rho)^2)\prod_{\rho'\neq\rho}\frac{\sigma(\rho)^2-\rho'^2}{\sigma(\rho)^2-\sigma(\rho')^2}.$$

If $\sigma=0\in\Sigma_K(u)$, identifying the term in x in both terms, one obtains $\|u'_0\|\neq 0$ if and only if $\displaystyle\prod_{\rho\in\Sigma_H(u)}\frac{\sigma(\rho)^2}{\rho^2}=0$, and in that case, identifying the corresponding factor of x gives

$$\|u'_0\|^2=\rho_{\max}^2\prod_{\sigma\in\Sigma_K(u)\setminus\{0\}}\frac{\rho(\sigma)^2}{\sigma^2}.$$

$\square$

As a consequence of the previous lemma, we get the following couple of corollaries.

Corollary 2. — *For any $\rho, \rho' \in \Sigma_H(u)$, $\sigma, \sigma' \in \Sigma_K(u) \setminus \{0\}$, we have*

$$\sum_{\rho \in \Sigma_H(u)} \frac{\|u_\rho\|^2}{\rho^2 - \sigma^2} = 1 \tag{3.2.10}$$

$$\sum_{\sigma \in \sigma_K(u)} \frac{\|u'_\sigma\|^2}{\rho^2 - \sigma^2} = 1 \tag{3.2.11}$$

$$\sum_{\rho \in \Sigma_H(u)} \frac{\|u_\rho\|^2}{(\rho^2 - \sigma^2)(\rho^2 - \sigma'^2)} = \frac{1}{\|u'_\sigma\|^2}\delta_{\sigma\sigma'} \tag{3.2.12}$$

$$\sum_{\sigma \in \Sigma_K(u)} \frac{\|u'_\sigma\|^2}{(\sigma^2 - \rho^2)(\sigma^2 - \rho'^2)} = \frac{1}{\|u_\rho\|^2}\delta_{\rho\rho'} \tag{3.2.13}$$

Proof. — The first two equalities (3.2.10) and (3.2.11) are obtained by making $x = \frac{1}{\sigma^2}$ and $x = \frac{1}{\rho^2}$ respectively in formula (3.2.1) and formula (3.2.2). For equality (3.2.12) in the case $\sigma = \sigma'$, we first make the change of variable $y = 1/x$ in formula (3.2.1) then differentiate both sides with respect to y and make $y = \sigma^2$. Equality (3.2.13) in the case $\rho = \rho'$ follows by differentiating equation (3.2.2) and making $x = \frac{1}{\rho^2}$. Both equalities in the other cases follow directly respectively from equality (3.2.10) and equality (3.2.11). □

Corollary 3. — *The kernel of H_u is $\{0\}$ if and only if*

$$\prod_{\rho \in \Sigma_H(u)} \frac{\sigma(\rho)^2}{\rho^2} = 0 \ , \quad \prod_{\sigma \in \Sigma_K(u)} \frac{\sigma^2}{\rho(\sigma)^2} = \infty \ .$$

Proof. — By the first part of theorem 4 in [13] — which is independent of multiplicity assumptions— , the kernel of H_u is $\{0\}$ if and only if $1 \in \overline{\mathrm{Ran}}(H_u) \setminus \mathrm{Ran}(H_u)$, where $\mathrm{Ran}(H_u)$ denotes the range of H_u. On the other hand,

$$u = \sum_{\rho \in \Sigma_H(u)} u_\rho = \sum_{\rho \in \Sigma_H(u)} \frac{H_u(H_u(u_\rho))}{\rho^2} \ ,$$

hence the orthogonal projection of 1 onto $\overline{\mathrm{Ran}}(H_u)$ is

$$\sum_{\rho\in\Sigma_H(u)} \frac{H_u(u_\rho)}{\rho^2} .$$

Consequently, $1 \in \overline{\mathrm{Ran}}(H_u)$ if and only if

$$1 = \sum_{\rho\in\Sigma_H(u)} \left\| \frac{H_u(u_\rho)}{\rho^2} \right\|^2 = \sum_{\rho\in\Sigma_H(u)} \frac{\|u_\rho\|^2}{\rho^2} .$$

Moreover, if this is the case,

$$1 = \sum_{\rho\in\Sigma_H(u)} \frac{H_u(u_\rho)}{\rho^2}$$

and $1 \in \mathrm{Ran}(H_u)$ if and only if the series

$$\sum_{\rho\in\Sigma_H(u)} \frac{u_\rho}{\rho^2}$$

converges, which is equivalent to

$$\sum_{\rho\in\Sigma_H(u)} \frac{\|u_\rho\|^2}{\rho^4} < \infty .$$

Hence $1 \in \overline{\mathrm{Ran}}(H_u) \setminus \mathrm{Ran}(H_u)$ if and only if

$$\sum_{\rho\in\Sigma_H(u)} \frac{\|u_\rho\|^2}{\rho^2} = 1 , \quad \sum_{\rho\in\Sigma_H(u)} \frac{\|u_\rho\|^2}{\rho^4} = \infty ,$$

which is the claim, in view of identities (3.2.3) and (3.2.4). $\square$

3.3. Finite Blaschke products

In this section, we recall the definition and the structure of finite Blaschke products which appear in the spectral analysis of Hankel operators. A finite Blaschke product is an inner function of the form

$$\Psi(z) = \mathrm{e}^{-i\psi} \prod_{j=1}^{k} \chi_{p_j}(z) , \ \psi \in \mathbb{T} , \ p_j \in \mathbb{D} , \ \chi_p(z) := \frac{z-p}{1-\overline{p}z} , \ p \in \mathbb{D} .$$

The integer k is called the degree of Ψ. We denote by $\mathcal{B}_k$ the set of Blaschke products of degree k.

Alternatively, $\Psi \in \mathcal{B}_k$ can be written as

$$\Psi(z) = \mathrm{e}^{-i\psi} \frac{P(z)}{z^k \overline{P}\left(\frac{1}{z}\right)} \ ,$$

where $\psi \in \mathbb{T}$ is called the angle of Ψ and P is a monic polynomial of degree k with all its roots in $\mathbb{D}$. Such polynomials are called Schur polynomials. We denote by $\mathcal{O}_d$ the open subset of $\mathbb{C}^d$ made of $(a_1, \dots, a_d)$ such that $P(z) = z^d + a_1 z^{d-1} + \dots + a_d$ is a Schur polynomial. The following result implies that $\mathcal{B}_k$ is diffeomorphic to $\mathbb{T} \times \mathbb{R}^{2k}$. It is connected to the Schur–Cohn criterion [43], [8], and is classical in control theory, see *e.g.* [27] and references therein. For the sake of completeness, we give a self contained proof.

Proposition 2. — *For every $d \geq 1$ and $(a_1, \dots, a_d) \in \mathbb{C}^d$, the following two assertions are equivalent.*

1. $(a_1, \dots, a_d) \in \mathcal{O}_d$.
2. $|a_d| < 1$ *and*

$$\left(\frac{a_k - a_d \overline{a}_{d-k}}{1 - |a_d|^2}\right)_{1 \leq k \leq d-1} \in \mathcal{O}_{d-1} \ .$$

In particular, for every $d \geq 0$, $\mathcal{O}_d$ is diffeomorphic to $\mathbb{R}^{2d}$.

Proof. — Consider the rational functions

$$\chi(z) = \frac{z^d + a_1 z^{d-1} + \dots + a_d}{1 + \overline{a}_1 z + \dots + \overline{a}_d z^d} \ ,$$

and

$$\tilde{\chi}(z) = \frac{\chi(z) - \chi(0)}{1 - \overline{\chi(0)}\chi(z)} = z \frac{z^{d-1} + b_1 z^{d-2} + \dots + b_{d-1}}{1 + \overline{b}_1 z + \dots + \overline{b}_{d-1} z^{d-1}} \ , \ b_k := \frac{a_k - a_d \overline{a}_{d-k}}{1 - |a_d|^2} \ .$$

If (1) holds true, then $\chi \in \mathcal{B}_d$, which implies

$$\forall z \in \mathbb{D}, |\chi(z)| < 1 \ , \ |\chi(\mathrm{e}^{ix})| = 1 \ . \tag{3.3.1}$$

In particular, $\chi(0) = a_d \in \mathbb{D}$, and therefore the numerator and the denominator of $\tilde{\chi}$ have no common root. Moreover,

$$\forall z \in \mathbb{D}, |\tilde{\chi}(z)| < 1 \ , \ |\tilde{\chi}(\mathrm{e}^{ix})| = 1 \ . \tag{3.3.2}$$

This implies $\tilde{\chi} \in \mathcal{B}_d$, hence (2). Conversely, if (2) holds, then $\tilde{\chi}$ satisfies (3.3.2) and has degree d, hence

$$\chi(z) = \frac{\tilde{\chi}(z) + a_d}{1 + \overline{a}_d \tilde{\chi}(z)}$$

satisfies (3.3.1) and has degree d, whence (1).

The second statement follows from an easy induction argument on d, since $\mathcal{O}_1 = \mathbb{D}$ is diffeomorphic to $\mathbb{R}^2$. □

3.4. Two results by Adamyan–Arov–Krein

We recall the proof of two important results by Adamyan–Arov–Krein. The proof is translated from [1] into our representation of Hankel operators, and is given for the convenience of the reader.

Theorem 3.4.1 **(Adamyan, Arov, Krein [1]).** — *Let $u \in VMO_+(\mathbb{S}^1) \setminus \{0\}$. Denote by $(\lambda_k(u))_{k\geq 0}$ the sequence of singular values of H_u, namely the eigenvalues of $|H_u| := \sqrt{H_u^2}$, in decreasing order, and repeated according to their multiplicity. Let $k \geq 0, m \geq 1$, such that*

$$\lambda_{k-1}(u) > \lambda_k(u) = \cdots = \lambda_{k+m-1}(u) = s > \lambda_{k+m}(u) \ ,$$

with the convention $\lambda_{-1}(u) := +\infty$.

1. *For every $h \in E_u(s) \setminus \{0\}$, there exists a polynomial $P \in \mathbb{C}_{m-1}[z]$ such that*

$$\forall z \in \mathbb{D} \ , \ \frac{sh(z)}{H_u(h)(z)} = \frac{P(z)}{z^{m-1}\overline{P}\left(\frac{1}{z}\right)} \ .$$

2. *There exists a rational function r with no pole on $\overline{\mathbb{D}}$ such that $\mathrm{rk}(H_r) = k$ and*

$$\|H_u - H_r\| = s \ .$$

Proof. — We start with the case $k = 0$. In this case the statement (2) is trivial, so we just have to prove (1). This is a consequence of the following lemma.

Lemma 5. — *Assume $s = \|H_u\|$. For every $h \in E_u(s) \setminus \{0\}$, consider the following inner outer decompositions,*

$$h = ah_0 \ , \ s^{-1}H_u(h) = bf_0 \ .$$

If c is an arbitrary inner divisor of ab, $ab = cc'$, then $ch_0 \in E_u(s)$, with

$$H_u(ch_0) = sc'f_0 \ , \ H_u(c'f_0) = sch_0 \ . \tag{3.4.1}$$

In particular, a, b are finite Blaschke products and

$$\deg(a) + \deg(b) + 1 \leq \dim E_u(s) \ . \tag{3.4.2}$$

Furthermore, there exists an outer function h_0 such that, if $m :=$ $\dim E_u(s)$,

$$E_u(s) = \mathbb{C}_{m-1}[z]h_0 \ , \tag{3.4.3}$$

and there exists $\varphi \in \mathbb{T}$ such that, for every $P \in \mathbb{C}_{m-1}[z]$,

$$H_u(Ph_0)(z) = se^{i\varphi}z^{m-1}\overline{P}\left(\frac{1}{z}\right)h_0(z) \ . \tag{3.4.4}$$

Let us prove this lemma. We need a number of elementary properties of Toeplitz operators T_b defined by equation (2.2.1), where b is a function in $L^\infty_+ := L^2_+ \cap L^\infty$, which we recall below. In what follows, $u \in BMO_+$.

1.
$$H_uT_b = T_{\overline{b}}H_u = H_{T_{\overline{b}}u} \ .$$

2. If $|b| \leq 1$ on $\mathbb{S}^1$,
$$H_u^2 \geq T_{\overline{b}}H_u^2T_b \ .$$

3. If $|b| = 1$ on $\mathbb{S}^1$, namely b is an inner function,
$$\forall f \in L^2_+ \ , \ f = T_bT_{\overline{b}}f \Longleftrightarrow \|f\| = \|T_{\overline{b}}f\| \ .$$

Indeed, (1) is just equivalent to the elementary identities

$$\Pi(u\overline{bh}) = \Pi(\overline{b}\Pi(u\overline{h})) = \Pi(\Pi(\overline{b}u)\overline{h}) \ .$$

As for (2), we observe that $T_b^* = T_{\overline{b}}$ and

$$\|T_{\overline{b}}h\| \leq \|\overline{b}h\| \leq \|h\| \ .$$

Hence, using (1),

$$(H_u^2 h|h) - (T_{\overline{b}} H_u^2 T_b h|h) = \|H_u(h)\|^2 - \|T_{\overline{b}} H_u(h)\|^2 \geq 0 \ .$$

Finally, for (3) we remark that, if b is inner, $T_{\overline{b}} T_b = I$ and $T_b T_{\overline{b}}$ is the orthogonal projector onto the range of T_b, namely bL^2_+. Since $\|T_{\overline{b}} f\| = \|T_b T_{\overline{b}} f\|$, (3) follows.

Let us come back to the proof of Lemma 5. Starting from

$$H_u(h) = sf \ , \ H_u(f) = sh \ , \ h = ah_0 \ , \ f = bf_0 \ , \ ab = cc' \ ,$$

we obtain, using property (1),

$$T_{\overline{c'}} H_u(ch_0) = H_u(cc'h_0) = T_{\overline{b}} H_u(h) = sf_0 \ .$$

In particular,

$$\|H_u(ch_0)\| \geq \|T_{\overline{c'}} H_u(ch_0)\| = s\|f_0\| = s\|f\| = s\|h\| = s\|ch_0\| \ .$$

Since $s = \|H_u\|$, all the above inequalities are equalities, hence $ch_0 \in E_u(s)$, and, using (3),

$$H_u(ch_0) = T_{c'} T_{\overline{c'}} H_u(ch_0) = sc' f_0 \ .$$

The second identity in (3.4.1) immediately follows. Remark that, if Ψ is an inner function of degree at least d, there exist $d+1$ linearly independent inner divisors of Ψ in L^∞_+. Then inequality (3.4.2) follows. Let us come to the last part. Since $\dim E_u(s) = m$, there exists $h \in E_u(s) \setminus \{0\}$ such that the first $m-1$ Fourier coefficients of h cancel, namely

$$h = z^{m-1} \tilde{h} \ .$$

Considering the inner outer decompositions

$$\tilde{h} = ah_0 \ , \ H_u(h) = sbf_0 \ ,$$

and applying the first part of the lemma, we conclude that $\deg(a) + \deg(b) = 0$, hence, up to a slight change of notation, $a = b = 1$, and, for $\ell = 0, 1, \dots, m-1$,

$$H_u(z^\ell h_0) = sz^{m-1-\ell} f_0 \ , \ H_u(z^{m-\ell-1} f_0) = sz^\ell h_0 \ .$$

This implies

$$E_u(s) = \mathbb{C}_{m-1}[z] h_0 = \mathbb{C}_{m-1}[z] f_0 \ .$$

Since $\|h_0\| = \|h\| = \|f\| = \|f_0\|$, it follows that $f_0 = \mathrm{e}^{i\varphi} h_0$, and (3.4.4) follows from the antilinearity of H_u. The proof of Lemma 5 is complete.

Let us complete the proof of the theorem by proving the case $s < \|H_u\|$. The crucial observation is the following.

Lemma 6. — *There exists a function $\phi \in L^\infty$ such that $|\phi| = 1$ on $\mathbb{S}^1$, and such that the operators H_u and $H_{s\Pi(\phi)}$ coincide on $E_u(s)$.*

Let us prove this lemma. For every pair (h, f) of elements of $E_u(s)$ such that $H_u(h) = sf, H_u(f) = sh$, we claim that the function

$$\phi := \frac{f}{\overline{h}} ,$$

does not depend on the choice of the pair (h, f). Indeed, it is enough to check that, if (h', f') is another such pair,

$$f\overline{h}' = f'\overline{h} . \tag{3.4.5}$$

In fact, for every $n \geq 0$,

$$\begin{aligned} s(f\overline{h}'|z^n) &= (H_u(h)|S^n h') = ((S^*)^n H_u(h)|h') \\ &= (H_u(S^n h)|h') = (H_u(h')|S^n h) = s(f'\overline{h}|z^n) . \end{aligned}$$

Changing the role of (h, h') and (f, f'), we get the claim. Finally the fact $|\phi| = 1$ comes from applying the above identity to the pairs (h, f) and (f, h). Then we just have to check that, for every such pair,

$$H_{s\Pi(\phi)}(h) = s\Pi(\Pi(\phi)\overline{h}) = s\Pi(\phi\overline{h}) = sf .$$

This completes the proof of Lemma 6.

Let us first consider part (2) of the Theorem. Introduce

$$v := s\Pi(\phi) .$$

We are going to show that $r := u - v$ is a rational function with no pole on $\overline{\mathbb{D}}$, $\mathrm{rk}(H_r) = k$ and

$$\|H_u - H_r\| = s .$$

Since, for every $h \in L^2_+$,

$$H_v(h) = s\Pi(\phi\overline{h}) ,$$

we infer $\|H_v\| \le s$, and from $E_u(s) \subset E_v(s)$, we conclude

$$\|H_v\| = s \ .$$

Because of (2.1.3), H_u and H_v coincide on the smallest shift invariant closed subspace of L^2_+ containing $E_u(s)$. By Beurling's theorem [5], this subspace is aL^2_+ for some inner function a. Then $H_r = 0$ on aL^2_+, hence the rank of H_r is at most the dimension of $(aL^2_+)^\perp$. Since

$$\|H_u - H_r\| = \|H_v\| = s < \lambda_{k-1}(u) \ ,$$

the rank of H_r cannot be smaller than k, and the result will follow by proving that the dimension of $(aL^2_+)^\perp$ is k.

We can summarize the above construction as

$$H_{T_{\overline{a}}u} = H_u T_a = H_v T_a = H_{T_{\overline{a}}v} \ .$$

The above Hankel operator is compact and its norm is at most s. In fact, if $H_u(h) = sf \ , \ H_u(f) = sh$, with $f = a\tilde{f}$, it is clear from property (1) above that

$$H_{T_{\overline{a}}u}(h) = s\tilde{f} \ , \ H_{T_{\overline{a}}u}(\tilde{f}) = sh \ .$$

In particular,

$$\|H_{T_{\overline{a}}u}\| = s \ .$$

Applying property (3.4.1) from Lemma 5, we conclude that there exists an outer function h_0 such that

$$ch_0 \in E_{T_{\overline{a}}u}(s)$$

for every inner divisor c of a. Moreover, a is a Blaschke product of finite degree d. Since h_0 is outer, it does not vanish at any point of $\mathbb{D}$, therefore, it is easy to find d inner divisors $c_1, \dots, c_d$ of a such that $c_1h_0, \dots, c_dh_0$ are linearly independent and generate a vector subspace $\tilde{E}$ satisfying

$$\tilde{E} \cap aL^2_+ = \{0\} \ .$$

Consequently, we obtain

$$\tilde{E} \oplus E_u(s) \subset E_{T_{\overline{a}}u}(s),$$

whence

$$d' := \dim E_{T_{\overline{a}}u}(s) \ge d + m \ . \tag{3.4.6}$$

On the other hand, by property (2) above, we have

$$H^2_{T_{\overline{a}}u} = T_{\overline{a}} H_u^2 T_a \le H_u^2 \ ,$$

therefore, from the min-max formula,

$$\forall n, \lambda_n(T_{\overline{a}}u) \le \lambda_n(u) \ .$$

In particular, from the definition of d',

$$s = \lambda_{d'-1}(T_{\overline{a}}u) \le \lambda_{d'-1}(u) \ ,$$

which imposes, in view of the assumption, $d'-1 \le k+m-1$, in particular, in view of (3.4.6),

$$d \le k \ .$$

Finally, notice that, since a has degree d, the dimension of $(aL^2_+)^\perp$ is d. Hence, by the min-max formula again,

$$\lambda_d(u) \le \sup_{h\in aL^2_+\setminus\{0\}} \frac{\|H_u(h)\|}{\|h\|} \le s < \lambda_{k-1}(u) \ .$$

This imposes $d \ge k$, and finally

$$d = k \ , \ d' = k+m \ ,$$

and part (2) of the theorem is proved.

In order to prove part (1), we apply properties (3.4.3) and (3.4.4) of Lemma 5. We describe elements of $E_{T_{\overline{a}}u}(s)$ as

$$h(z) = Q(z)h_0(z) \ ,$$

where h_0 is outer and $Q \in \mathbb{C}_{k+m-1}[z]$. Moreover, if $h \in E_u(s)$, then $h = a\tilde{h}$, $H_u(h) = sa\tilde{f}$, where $\tilde{h}, \tilde{f} \in E_{T_{\overline{a}}u}(s)$. This reads

$$Q(z) = a(z)\tilde{Q}(z) \ ,$$

If we set

$$a(z) = \frac{z^k\overline{D}\left(\frac{1}{z}\right)}{D(z)} \ ,$$

where $D \in \mathbb{C}_k[z]$ and $D(0) = 1$, and D has no zeroes in $\mathbb{D}$, this implies

$$Q(z) = z^k\overline{D}\left(\frac{1}{z}\right) P(z) \ , \ P \in \mathbb{C}_{m-1}[z] \ .$$

Moreover,

$$H_{T_{\overline{a}}u}(h)(z) = s\mathrm{e}^{i\varphi}z^{m+k-1}\overline{Q}\left(\frac{1}{z}\right)h_0(z) = s\mathrm{e}^{i\varphi}D(z)z^{m-1}\overline{P}\left(\frac{1}{z}\right)h_0(z)\ ,$$

and

$$H_u(h)(z) = a(z)H_{T_{\overline{a}}u}(h)(z) = s\mathrm{e}^{i\varphi}z^k\overline{D}\left(\frac{1}{z}\right)z^{m-1}\overline{P}\left(\frac{1}{z}\right)h_0(z)\ .$$

Changing P into $P\mathrm{e}^{-i\varphi/2}$, this proves part (1) of the theorem. □

3.5. Multiplicity and Blaschke products

In this section, we prove a refinement of part (1) of Theorem 3.4.1 which allows to describe precisely the structure of the eigenspaces of a Hankel operator and its shifted. For a Blaschke product $\Psi \in \mathcal{B}_k$ given by

$$\Psi(z) = \mathrm{e}^{-i\psi}\frac{P(z)}{z^k\overline{P}\left(\frac{1}{z}\right)},$$

we shall denote by

$$D(z) = z^k\overline{P}\left(\frac{1}{z}\right)$$

the normalized denominator of Ψ.
Assume $u \in VMO_+(\mathbb{S}^1)$ and $s \in \Sigma_H(u)$. Then H_u acts on the finite dimensional vector space $E_u(s)$. It turns out that this action can be completely described by an inner function. A similar fact holds for the action of K_u onto $F_u(s)$ when $s \in \Sigma_K(u)$, $s \neq 0$.

Proposition 3. — *Let $s > 0$ and $u \in VMO_+(\mathbb{S}^1)$.*

1. *Assume $s \in \Sigma_H(u)$ and $m := \dim E_u(s) = \dim F_u(s) + 1$. Denote by u_s the orthogonal projection of u onto $E_u(s)$. There exists an inner function $\Psi_s \in \mathcal{B}_{m-1}$ such that*

$$su_s = \Psi_s H_u(u_s)\ , \tag{3.5.1}$$

and if D denotes the normalized denominator of Ψ_s,

$$E_u(s) = \left\{\frac{f}{D}H_u(u_s)\ ,\ f \in \mathbb{C}_{m-1}[z]\right\}\ , \tag{3.5.2}$$

$$F_u(s) = \left\{\frac{g}{D}H_u(u_s)\ ,\ g \in \mathbb{C}_{m-2}[z]\right\}, \tag{3.5.3}$$

and, for $a = 0, \dots, m-1$, $b = 0, \dots, m-2$,

$$H_u\left(\frac{z^a}{D}H_u(u_s)\right) = se^{-i\psi_s}\frac{z^{m-a-1}}{D}H_u(u_s) \ , \tag{3.5.4}$$

$$K_u\left(\frac{z^b}{D}H_u(u_s)\right) = se^{-i\psi_s}\frac{z^{m-b-2}}{D}H_u(u_s) \ , \tag{3.5.5}$$

where ψ_s *denotes the angle of* Ψ_s.

2. *Assume* $s \in \Sigma_K(u)$ *and* $\ell := \dim F_u(s) = \dim E_u(s) + 1$. *Denote by* u_s' *the orthogonal projection of* u *onto* $F_u(s)$. *There exists an inner function* $\Psi_s \in \mathcal{B}_{\ell-1}$ *such that*

$$K_u(u_s') = s\Psi_s u_s' \ , \tag{3.5.6}$$

and if D *denotes the normalized denominator of* Ψ_s,

$$F_u(s) = \left\{\frac{f}{D}u_s' \ , \ f \in \mathbb{C}_{\ell-1}[z]\right\} \ , \tag{3.5.7}$$

$$E_u(s) = \left\{\frac{zg}{D}u_s' \ , \ g \in \mathbb{C}_{\ell-2}[z]\right\} , \tag{3.5.8}$$

and, for $a = 0, \dots, \ell-1$, $b = 0, \dots, \ell-2$,

$$K_u\left(\frac{z^a}{D}u_s'\right) = se^{-i\psi_s}\frac{z^{\ell-a-1}}{D}u_s' \ , \tag{3.5.9}$$

$$H_u\left(\frac{z^{b+1}}{D}u_s'\right) = se^{-i\psi_s}\frac{z^{\ell-b-1}}{D}u_s' \ , \tag{3.5.10}$$

where ψ_s *denotes the angle of* Ψ_s.

3.5.1. Case of $\rho \in \Sigma_H(u)$. — Let $u \in VMO_+(\mathbb{S}^1)$. Assume that $\rho \in \Sigma_H(u)$ and $m := \dim E_u(\rho)$. Recall that

$$E_u(\rho) = \ker(H_u^2 - \rho^2 I), \ F_u(\rho) = \ker(K_u^2 - \rho^2 I)$$

and by Lemma 2,

$$F_u(\rho) = E_u(\rho) \cap u^\perp \ .$$

3.5.1.1. Definition of Ψ_ρ.— By definition $(\rho u_\rho, H_u(u_\rho)$ and $(H_u(u_\rho), \rho u_\rho)$ are Schmidt-pair for H_u. Hence, applying equation (3.4.5) to $(u_\rho, H_u(u_\rho))$ and $(H_u(u_\rho), u_\rho)$, we obtain that, at every point of $\mathbb{S}^1$,

$$|H_u(u_\rho)|^2 = \rho^2 |u_\rho|^2 \ .$$

We thus define

$$\Psi_\rho := \frac{\rho u_\rho}{H_u(u_\rho)} \ .$$

3.5.1.2. The function Ψ_ρ *is an inner function.*— We know that Ψ_ρ is of modulus 1 at every point of $\mathbb{S}^1$. Let us show that Ψ_ρ is in fact an inner function. By part (1) of the Adamyan–Arov–Krein theorem, we already know that Ψ_ρ is a rational function with no poles on the unit circle. Therefore, it is enough to prove that Ψ_ρ has no pole in the open unit disc. Assume that $q \in \mathbb{D}$ is a zero of $H_u(u_\rho)$, and let us show that q is a zero of u_ρ with at least the same multiplicity.

Assume $H_u(u_\rho)(q) = 0$ for some q, $|q| < 1$. We prove that $u_\rho(q) = 0$ (in fact $h(q) = 0$ for any h in $E_u(\rho)$). Let us consider the continuous linear form

$$\begin{array}{ccc} E_u(\rho) & \to & \mathbb{C} \\ h & \mapsto & h(q) \end{array} .$$

By the Riesz representation Theorem, there exists $f \in E_u(\rho)$ such that $h(q) = (h|f)$. Our aim is to prove that, if $H_u(u_\rho)(q) = 0$ then $f = 0$. The assumption $H_u(u_\rho)(q) = 0$ reads

$$0 = (H_u(u_\rho)|f) = (H_u(f)|u_\rho) = (H_u(f)|u) = (1|H_u^2(f)) = \rho^2(1|f).$$

Here we used the fact that, as $H_u(f) \in E_u(\rho)$ for any $f \in E_u(\rho)$,

$$(H_u(f)|u_\rho) = (H_u(f)|u).$$

It implies that $H_u(f) \in u^\perp \cap E_u(\rho) = F_u(\rho)$, that $S^*f = \frac{1}{\rho^2} S^* H_u^2 f = \frac{1}{\rho^2} K_u H_u(f)$ belongs to $F_u(\rho)$ and also $f = SS^*f$. Hence,

$$\|f\|^2 = (f|f) = f(q) = qS^*f(q) = q(S^*f|f) \le |q| \|S^*f\| \cdot \|f\|$$

which is impossible except if $f = 0$. In particular $u_\rho(q) = 0$.

A similar argument allows to show that if q is a zero of order r of $H_u(u_\rho)$, that is, if, for every $a \le r - 1$,

$$(H_u(u_\rho))^{(a)}(q) = 0$$

then the same holds for u_ρ that is $u_\rho^{(a)}(q) = 0$ for every $a \leq r-1$.

As a consequence of the Adamyan–Arov–Krein theorem 3.4.1, we get that Ψ_ρ is a Blaschke product of degree $m-1$, $m = \dim E_u(\rho)$.

3.5.1.3. Description of $E_u(s)$, $F_u(s)$ and of the action of H_u and of K_u on them. — We start with proving the following lemma.

Lemma 7. — *Let $f \in \mathbb{H}^\infty(\mathbb{D})$ such that $\Pi(\Psi_\rho \overline{f}) = \Psi_\rho \overline{f}$. Then*

$$H_u(fH_u(u_\rho)) = \rho \Psi_\rho \overline{f} H_u(u_\rho) \ .$$

The proof of the lemma is straightforward,

$$\begin{aligned} H_u(fH_u(u_\rho)) &= \Pi(u\overline{fH_u(u_\rho)}) = \Pi(\overline{f}H_u^2(u_\rho)) = \rho^2\Pi(\overline{f}u_\rho) = \rho\Pi(\overline{f}\Psi_\rho H_u(u_\rho)) \\ &= \rho\Psi_\rho\overline{f}H_u(u_\rho) \ . \end{aligned}$$

Applying this lemma, we get, for any $0 \leq a \leq m-1$,

$$H_u\left(\frac{z^a}{D}H_u(u_\rho)\right) = \rho \mathrm{e}^{-i\psi}\frac{z^{m-1-a}}{D}H_u(u_\rho) \ .$$

Consequently, $E_u(\rho) = \dfrac{\mathbb{C}_{m-1}[z]}{D}H_u(u_\rho)$ and that the action of H_u on $E_u(\rho)$ is as expected in Equation (3.5.4). It remains to prove that

$$F_u(\rho) = \frac{\mathbb{C}_{m-2}[z]}{D}H_u(u_\rho)$$

and that the action of K_u is described as in (3.5.4). We have, for $0 \leq b \leq m-2$,

$$\begin{aligned} K_u\left(\frac{z^b}{D}H_u(u_\rho)\right) &= H_uS\left(\frac{z^b}{D}H_u(u_\rho)\right) = H_u\left(\frac{z^{b+1}}{D}H_u(u_\rho)\right) \\ &= \rho\mathrm{e}^{-i\psi}\frac{z^{m-2-b}}{D}H_u(u_\rho) \end{aligned}$$

In particular, it proves that $\frac{\mathbb{C}_{m-2}[z]}{D}H_u(u_\rho) \subset F_u(\rho)$. As the dimension of $F_u(\rho)$ is $m-1$ by assumption, we get the equality.

3.5.2. Case of $\sigma \in \Sigma_K(u)$. — The second part of the proposition, concerning the case of $\sigma \in \Sigma_K(u)$, can be proved similarly. The first step is to prove that

$$\Psi_\sigma := \frac{K_u(u'_\sigma)}{\sigma u'_\sigma}$$

is an inner function. The same argument as the one used above for the K_u Schmidt pairs $(K_u(u'_\sigma), \sigma u'_\sigma)$ and $(\sigma u'_\sigma, K_u(u'_\sigma))$ gives that Ψ_σ has modulus one. To prove that it is an inner function, we argue as before. Namely, using again part (1) of the Adamyan-Arov–Krein theorem 3.4.1, for S^*u in place of u, we prove that if u'_σ vanishes at some $q \in \mathbb{D}$, $K_u(u'_\sigma)$ also vanishes at q at the same order.

Assume $u'_\sigma(q) = 0$. As before, there exists $f \in F_u(\sigma)$ satisfying $h(q) = (h|f)$ for any $h \in F_u(\sigma)$. The assumption on u'_σ reads $0 = (u'_\sigma|f) = (u|f)$ hence f belongs to $E_u(\sigma) = u^\perp \cap F_u(\sigma)$. The same holds for $H_u(f)$ and, in particular, $0 = (H_u(f)|u) = (1|H_u^2(f)) = \sigma^2(1|f)$. Therefore, $f = SS^*f$ and $S^*f = K_uH_u(f) \in F_u(\sigma)$. One concludes as before since

$$\|f\|^2 = f(q) = qS^*f(q) = q(S^*f|f)$$

which implies $f = 0$.

One proves as well that if q is a zero of order r of (u'_σ), $K_u(u'_\sigma)$ vanishes at q with the same order. Eventually, by theorem 3.4.1, $\Psi_\sigma \in \mathcal{B}_{\ell-1}$. It remains to describe $E_u(\sigma)$ and $F_u(\sigma)$ and the action of H_u and K_u on them. We start with a lemma analogous to Lemma 7.

Lemma 8. — *Let $f \in \mathbb{H}^\infty(\mathbb{D})$ such that $\Pi(\Psi_\sigma \overline{f}) = \Psi_\sigma \overline{f}$. Then*

$$K_u(fu'_\sigma) = \sigma \Psi_\sigma \overline{f} u'_\sigma \ .$$

The proof of the lemma is similar to the one of Lemma 7. Write

$$\Psi_\sigma(z) = \mathrm{e}^{-i\psi} \frac{z^{\ell-1}\overline{D}\left(\frac{1}{z}\right)}{D(z)},$$

where D is some normalized polynomial of degree k. From the above lemma, for $0 \le a \le \ell - 1$,

$$K_u\left(\frac{z^a}{D} u'_\sigma\right) = \sigma \mathrm{e}^{-i\psi} \frac{z^{\ell-1-a}}{D} u'_\sigma \ .$$

In order to complete the proof, we just need to describe $E_u(\sigma)$ as the subspace of $F_u(\sigma)$ made with functions which vanish at $z = 0$, or equivalently are orthogonal to 1. We already know that vectors of $E_u(\sigma)$ are orthogonal to u, and that H_u is a bijection from $E_u(\sigma)$ onto $E_u(\sigma)$. We infer that vectors of $E_u(\sigma)$ are orthogonal to 1. A dimension argument allows to conclude.

CHAPTER 4

THE INVERSE SPECTRAL THEOREM

We now come to the analysis of the non-linear Fourier transform introduced in Theorem 2. Given $u \in H^{1/2}_+(\mathbb{S}^1) \setminus \{0\}$, one can define, according to Lemma 3, a finite or infinite sequence $s = (s_1 > s_2 > \dots)$ such that

1. The s_{2j-1}'s are the H-dominant singular values associated to u.
2. The s_{2k}'s are the K-dominant singular values associated to u.

For every $r \geq 1$, associate to each s_r a Blaschke product Ψ_r by means of Proposition 3. This defines a mapping

$$\Phi : H^{1/2}_+(\mathbb{S}^1) \setminus \{0\} \longrightarrow \mathcal{S}^{(2)}_\infty \cup \bigcup_{n=1}^{\infty} \mathcal{S}_n \ .$$

We are going to prove that Φ is bijective and coincides with the inverse mapping of the non linear Fourier transform introduced in Theorem 2. In other words, with the notation of Theorem 2, $\Phi(u) = (\mathbf{s}, \boldsymbol{\Psi}) \in \mathcal{S}_n$ if and only if

$$u = u(\mathbf{s}, \boldsymbol{\Psi}) \tag{4.0.11}$$

Furthermore, if $(s_r, \Psi_r)_{r\geq 1} \in \mathcal{S}^{(2)}_\infty$, then its preimage u by Φ is given by

$$u = \lim_{q\to\infty} u_q \tag{4.0.12}$$

where the limit takes place in $H^{\frac{1}{2}}_+(\mathbb{S}^1)$ and

$$u_q := u((s_1, \dots, s_{2q}), (\Psi_1, \dots, \Psi_{2q})).$$

Let us briefly describe the structure of this chapter. Firstly, we concentrate on the case of finite rank operators. This part itself includes two

important steps. The first step is to prove injectivity of Φ by producing an explicit formula for u, which coincides with the one for $u(\mathbf{s}, \mathbf{\Psi})$ in Theorem 2 of the Introduction. The second step establishes surjectivity by combining algebraic calculations and compactness arguments. This step includes in particular the construction of Hankel operators with arbitrary multiplicities thanks to a process based on collapsing three consecutive singular values. Section 4.2 extends these results to Hilbert–Schmidt operators, and section 4.3 to compact Hankel operators.

4.1. The inverse spectral theorem in the finite rank case

In this section, we consider the case of finite rank Hankel operators. Consider a symbol u with H_u of finite rank. Then the sets $\Sigma_H(u)$ and $\Sigma_K(u)$ are finite. We set

$$q := |\Sigma_H(u)| = |\Sigma_K(u)| \ .$$

If

$$\Sigma_H(u) := \{\rho_j, j = 1, \dots, q\},\ \rho_1 > \cdots > \rho_q > 0 \ ,$$
$$\Sigma_K(u) := \{\sigma_j, j = 1, \dots, q\},\ \sigma_1 > \cdots > \sigma_q \geq 0 \ ,$$

we know from Lemma (3) that

$$\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \cdots > \rho_q > \sigma_q \geq 0 \ . \tag{4.1.1}$$

We set $n := 2q$ if $\sigma_q > 0$ and $n := 2q - 1$ if $\sigma_q = 0$. For $2j \leq n$, we set

$$s_{2j-1} := \rho_j \ ,\ s_{2j} := \sigma_j \ ,$$

so that the positive elements in the list (4.1.1) read

$$s_1 > s_2 > \cdots > s_n > 0 \ . \tag{4.1.2}$$

Using Proposition 3, we define n finite Blaschke products $\Psi_1, \dots, \Psi_n$ by

$$\rho_j u_j = \Psi_{2j-1} H_u(u_j) \ ,\ K_u(u'_j) = \sigma_j \Psi_{2j} u'_j \ ,\ 2j \leq n \ ,$$

where u_j denotes the orthogonal projection of u onto $E_u(\rho_j)$, and u'_j denotes the orthogonal projection of u onto $F_u(\sigma_j)$.

Given a finite sequence $(d_1, \dots, d_n)$ of nonnegative integers, recall that $\mathcal{V}_{(d_1,\dots,d_n)}$ is the set of symbols u such that

1. The singular values s_{2j-1}, $j \geq 1$, of H_u in $\Sigma_H(u)$, ordered decreasingly, have respective multiplicities
$$d_1 + 1, d_3 + 1, \dots \ .$$
2. The singular values s_{2j}, $j \geq 1$, of K_u in $\Sigma_K(u)$, ordered decreasingly, have respective multiplicities
$$d_2 + 1, d_4 + 1, \dots \ .$$

Our goal is to prove the following statement.

Theorem 5. — *The mapping*

$$\begin{array}{rcl} \Phi_{d_1,\dots,d_n} : \mathcal{V}_{d_1,\dots,d_n} & \longrightarrow & \mathcal{S}_{d_1,\dots,d_n} \\ u & \longmapsto & ((s_r)_{1\leq r\leq n}, (\Psi_r)_{1\leq r\leq n}) \end{array}$$

is a homeomorphism and $\Phi_{d_1,\dots,d_n}(u) = (\mathbf{s}, \mathbf{\Psi})$ *is equivalent to*

$$u = u(\mathbf{s}, \mathbf{\Psi})$$

with the formula given by 1.0.5.
Moreover, the mapping $\Phi^{-1}_{d_1,\dots,d_n} : \mathcal{S}_{d_1,\dots,d_n} \to \mathcal{V}(d)$, *where* d *is given by (3.1.5), is a smooth immersion. As a consequence, the set* $\mathcal{V}_{(d_1,\dots,d_n)}$ *is a submanifold of* $\mathcal{V}(d)$ *of dimension*

$$\dim \mathcal{V}_{(d_1,\dots,d_n)} = 2n + 2\sum_{r=1}^{n} d_r \ .$$

Proof. — The proof of Theorem 5 involves several steps. Firstly, we prove the continuity of $\Phi_{d_1,\dots,d_n}$, and more precisely that every point of $\mathcal{V}_{(d_1,\dots,d_n)}$ has an open neihgborhood in $\mathcal{V}(d)$, where d is given by (3.1.5), on which $\Phi_{d_1,\dots,d_n}$ can be extended as a smooth mapping $\tilde{\Phi}_n$ into some finite dimensional manifold including $\mathcal{S}_{d_1,\dots,d_n}$. Then we prove that $\Phi_{d_1,\dots,d_n}$ is a homeomorphism in the case n even, along the following lines :

– $\Phi_{d_1,\dots,d_n}$ is injective, with the formula
$$\Phi_{d_1,\dots,d_n}(u) = (\mathbf{s}, \mathbf{\Psi}) \Rightarrow u = u(\mathbf{s}, \mathbf{\Psi}).$$
– $\Phi_{d_1,\dots,d_n}$ is an open mapping.
– $\Phi_{d_1,\dots,d_n}$ is a proper mapping.
– $\mathcal{V}_{(d_1,\dots,d_n)}$ is not empty.

Since the target space $\mathcal{S}_{(d_1,\dots,d_n)}$ is connected, these four items trivially lead to the homeomorphism result. The case n odd is derived from a simple limiting argument. Then the right inverse formula proved in the first item implies that, for every $(\mathbf{s},\mathbf{\Psi})\in\mathcal{S}_{d_1,\dots,d_n}$, $u(\mathbf{s},\mathbf{\Psi})\in\mathcal{V}_{(d_1,\dots,d_n)}$ and

$$\Phi_{d_1,\dots,d_n}(u(\mathbf{s},\mathbf{\Psi}))=(\mathbf{s},\mathbf{\Psi}).$$

The use of the smooth mapping $\tilde{\Phi}_n$ combined to the smoothness of the mapping $(\mathbf{s},\mathbf{\Psi})\mapsto u(\mathbf{s},\mathbf{\Psi})$ will then complete the proof.

4.1.1. Continuity of $\Phi_{d_1,\dots,d_n}$ and local smooth extension. — Fix $u_0\in\mathcal{V}_{(d_1,\dots,d_n)}$. Let $\rho\in\Sigma_H(u_0)$. The orthogonal projector P_ρ on the eigenspace $E_{u_0}(\rho)$ is given by

$$P_\rho=\int_{\mathscr{C}_\rho}(zI-H_{u_0}^2)^{-1}\frac{dz}{2i\pi}$$

where $\mathscr{C}_\rho$ is a circle, centered at ρ^2 whose radius is small enough so that the closed disc $\overline{D}_\rho$ delimited by $\mathscr{C}_\rho$ is at positive distance to the rest of the spectrum of $H_{u_0}^2$. For u in a neighborhood V_0 of u_0 in $H^{1/2}_+$, C_ρ does not meet the spectrum of H_u^2, and one may consider

$$P_\rho^{(u)}:=\int_{\mathscr{C}_\rho}(zI-H_u^2)^{-1}\frac{dz}{2i\pi}$$

which is a finite rank orthogonal projector smoothly dependent on u. Hence, $P_\rho^{(u)}(u)$ is well defined and smooth. Since this vector is not zero for $u=u_0$, it is still not zero for every u in V_0.

We can do the same construction with any $\sigma\in\Sigma_K(u_0)\setminus\{0\}$. We have therefore constructed n smooth functions $u\in V_0\mapsto P_r^{(u)}$, $r=1,\dots,n$, valued in the finite orthogonal projectors, and satisfying

$$P_r^{(u)}(u)\neq 0\ ,\ r=1,\dots,n\ .$$

Moreover, by continuity,

$$\mathrm{rk}P_r^{(u)}=\mathrm{rk}P_r^{(u_0)}:=d_r+1\ .$$

In the special case $u \in \mathcal{V}_{(d_1,\dots,d_n)}$, $P_r^{(u)}$ is precisely the orthogonal projector onto $E_u(s_r) + F_u(s_r)$. We then define a map $\tilde{\Phi}_n$ on $V_0 \cap \mathcal{V}(d)$ by setting

$$\tilde{\Phi}_n(u) = ((s_r(u))_{1\le r\le n}; (\Psi_r(u))_{1\le r\le n}) \ ,$$

with

$$s_{2j-1}(u) := \frac{\|H_u(P_{2j-1}^{(u)}(u))\|}{\|P_{2j-1}^{(u)}(u)\|} \quad , \quad s_{2k}(u) := \frac{\|K_u(P_{2k}^{(u)}(u))\|}{\|P_{2k}^{(u)}(u)\|} \ ,$$

$$\Psi_{2j-1}(u) := \frac{s_{2j-1}(u)P_{2j-1}^{(u)}(u)}{H_u(P_{2j-1}^{(u)}(u))} \quad , \quad \Psi_{2k}(u) = \frac{K_u(P_{2k}^{(u))}(u)}{s_{2k}(u)P_{2k}^{(u)}(u)} \ .$$

The mapping $\tilde{\Phi}_n$ is smooth from $V_0 \cap \mathcal{V}(d)$ into $\Omega_n \times \mathcal{R}_d^n$, where $\mathcal{R}_d$ denotes the manifold of rational functions with numerators and denominators of degree at most $\left[\frac{d+1}{2}\right]$. Moreover, from Proposition 3, the restriction of $\tilde{\Phi}_n$ to $V_0 \cap \mathcal{V}_{(d_1,\dots,d_n)}$ coincides with $\Phi_{d_1,\dots,d_n}$. This proves in particular that $\Phi_{d_1,\dots,d_n}$ is continuous. For future reference, let us state more precisely what we have proved.

Lemma 9. — *For every $u_0 \in \mathcal{V}_{(d_1,\dots,d_n)}$, there exists a neighborhood V of u_0 in $\mathcal{V}(d)$, $d = n + 2\sum_{r=1}^n d_r$, and a smooth mapping $\tilde{\Phi}_n$ from this neighborhood into some manifold, such that the restriction of $\tilde{\Phi}_n$ to $V \cap \mathcal{V}_{(d_1,\dots,d_n)}$ coincides with $\Phi_{d_1,\dots,d_n}$.*

4.1.2. The explicit formula, case n even.— Assume that $n = 2q$ is an even integer.

The fact that the mapping $\Phi_{d_1,\dots,d_n}$ is one-to-one follows from an explicit formula giving u in terms of $\Phi_{d_1,\dots,d_n}(u)$, which we establish in this subsection.

We use the expected description of elements of $\Phi^{-1}(\mathcal{S}_n)$ suggested by the action of H_u, K_u onto the orthogonal projections u_j, u'_k of u onto the corresponding eigenspaces of H_u^2, K_u^2 respectively, namely

$$\rho_j u_j = \Psi_{2j-1} H_u(u_j) \ , \ K_u(u'_k) = \sigma_k \Psi_{2k} u'_k \ , \ j,k = 1,\dots,q \ , \tag{4.1.3}$$

where the Ψ_r's are Blaschke products.

Using the identity $I = SS^* + (\,.\,|1)$, we obtain

$$H_u(u_j) = SK_u(u_j) + \|u_j\|^2 \ .$$

We then use the formula (3.1.6), in this setting

$$u_j = \tau_j^2 \sum_{k=1}^q \frac{1}{\rho_j^2 - \sigma_k^2} u'_k \ , \ \tau_j^2 = \|u_j\|^2 \ , \tag{4.1.4}$$

to get

$$\begin{aligned} \Psi_{2j-1}(z) H_u(u_j)(z) &= \tau_j^2 \Psi_{2j-1}(z) \left(z \sum_{k=1}^q \frac{K_u(u'_k)(z)}{\rho_j^2 - \sigma_k^2} + 1 \right) \\ &= \tau_j^2 \left(\sum_{k=1}^q \frac{\sigma_k z \Psi_{2j-1}(z) \Psi_{2k}(z) u'_k(z)}{\rho_j^2 - \sigma_k^2} + \Psi_{2j-1}(z) \right) \ , \end{aligned}$$

after using the second identity of (4.1.3). On the other hand, using again (4.1.4),

$$\rho_j u_j(z) = \tau_j^2 \sum_{j=1}^k \frac{\rho_j \, u'_k(z)}{\rho_j^2 - \sigma_k^2} \ .$$

Applying the first identity in (4.1.3), we conclude

$$\sum_{k=1}^q \frac{\rho_j \ - z\sigma_k z \Psi_{2j-1}(z)\Psi_{2k}(z)}{\rho_j^2 - \sigma_k^2} u'_k(z) = \Psi_{2j-1}(z) \ ,$$

which is precisely

$$\mathscr{C}(z)\mathcal{U}'(z) = (\Psi_{2j-1}(z))_{1\le j\le q} \ , \tag{4.1.5}$$

where

$$c_{k\ell}(z) := \frac{\rho_k - \sigma_\ell z \Psi_{2k-1}(z)\Psi_{2\ell}(z)}{\rho_k^2 - \sigma_\ell^2} \ , \tag{4.1.6}$$

and $\mathcal{U}'(z) := (u'_k(z))_{1\le k\le q}$. Since

$$u = \sum_{k=1}^q u'_k \ ,$$

we conclude, for every z such that $\mathscr{C}(z)$ is invertible,

$$u(z) = \langle \mathscr{C}(z)^{-1} (\Psi_{2j-1}(z))_{1\le j\le q}, \mathbf{1} \rangle \ ,$$

as claimed by formula (1.0.5).

Similarly, writing $H_u(u'_k) = SK_u(u'_k) + \|u'_k\|^2$, and using (3.1.7) in this setting which in this setting reads

$$u'_k = \kappa_k^2 \sum_{j=1}^{q} \frac{1}{\rho_j^2 - \sigma_k^2} u_j \ , \ \kappa_k^2 = \|u'_k\|^2 \ , \tag{4.1.7}$$

we obtain that

$$u_j = \Psi_{2j-1} h_j \ , \ \text{or } h_j := \frac{1}{\rho_j} H_u(u_j) \ ,$$

where

$$\mathcal{H}(z) := (h_j(z))_{1\le j\le q} \ ,$$

is the solution of

$$ {}^t\mathscr{C}(z)\mathcal{H}(z) = \mathbf{1} \ . \tag{4.1.8}$$

Let us come to the invertibility of matrices $\mathscr{C}(z)$. Since $\mathscr{C}(0)$ is invertible, $\mathscr{C}(z)$ is invertible for z in a small disc centered at 0. This is enough for proving the injectivity of $\Phi_{d_1,\dots,d_n}$. However, let us prove that this property holds for every z in the closed unit disc. Recall that

$$\Psi_r(z) = \mathrm{e}^{-i\psi_r} \frac{P_r(z)}{D_r(z)} \ , \ D_r(z) := z^{d_r} \overline{P}_r \left(\frac{1}{z}\right) \ , \tag{4.1.9}$$

where P_r is a monic polynomial of degree d_r. Introduce the matrix $\mathscr{C}^\#(z) = (c^\#_{k\ell}(z))_{1\le k,\ell\le q}$ as

$$c^\#_{k\ell}(z) = \frac{\rho_k D_{2\ell}(z) D_{2k-1}(z) - \sigma_\ell z \mathrm{e}^{-i(\psi_{2\ell}+\psi_{2k-1})} P_{2\ell}(z) P_{2k-1}(z)}{\rho_k^2 - \sigma_\ell^2} \ . \tag{4.1.10}$$

Introducing the notation

$$\mathrm{diag}(\lambda_j)_{1\le j\le q} := \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_q \end{pmatrix}$$

we have

$$\mathscr{C}(z) = \mathrm{diag}\left(\frac{1}{D_{2k-1}(z)}\right) \mathscr{C}^\#(z) \mathrm{diag}\left(\frac{1}{D_{2\ell}(z)}\right) \ ,$$

so that $u(z)$ can be written equivalently as

$$u(z) = \langle \mathscr{C}^\#(z)^{-1} (\mathrm{e}^{-i\psi_{2j-1}} P_{2j-1}(z))_{1\le j\le q}, (D_{2k}(z))_{1\le k\le q} \rangle \ ,$$

for every z such that $\det \mathscr{C}^{\#}(z) \neq 0$. Using Cramer's formulae for the inverse of a matrix, it is easy to see that the degree of the denominator of the above rational function is at most $N = N := q + \sum_{r=1}^{n} d_r$, and that the degree of the numerator is at most $N-1$. If $\det \mathscr{C}^{\#}(z) = 0$ for some z in the closed unit disc, then, in order to preserve the analyticity of u, u should be written as a quotient of polynomials of degrees smaller than $N-1$ and N respectively, so that the rank of H_u would be smaller than it is. Consequently, $\det \mathscr{C}^{\#}(z) \neq 0$ for every z in the close unit disc, so that formula (1.0.5) holds for every such z.

4.1.3. Surjectivity in the case n even.— Our purpose is now to prove that the mapping $\Phi_{d_1,\dots,d_n}$ is onto. Since we got a candidate from the formula giving u in the latter section, it may seem natural to try to check that this formula indeed provides an element u of $\mathcal{V}_{(d_1,\dots,d_n)}$ with the required $\Phi_{d_1,\dots,d_n}(u)$. However, in view of the complexity of formula (1.0.5) , it seems difficult to infer from them the spectral properties of H_u and K_u. We shall therefore use an indirect method, by proving that the mapping $\Phi_{d_1,\dots,d_n}$ on $\mathcal{V}_{(d_1,\dots,d_n)}$ is open, closed, and that the source space $\mathcal{V}_{(d_1,\dots,d_n)}$ is not empty. Since the target space $\mathcal{S}_{(d_1,\dots,d_n)}$ is clearly connected, this will imply the surjectivity. A first step in proving that $\Phi_{d_1,\dots,d_n}$ is an open mapping, consists in the construction of antilinears operators H and K satisfying the required spectral properties, and which will be finally identified as H_u and K_u.

4.1.3.1. Construction of the operators H and K. — Let

$$\mathcal{P} = ((s_r)_{1\le r\le n}, (\Psi_r)_{1\le r\le n})$$

be an arbitrary element of $\mathcal{S}_{(d_1,\dots,d_n)}$ for some non negative integers d_r . We look for $u \in \mathcal{V}_{(d_1,\dots,d_n)}$, $\Phi_{d_1,\dots,d_n}(u) = \mathcal{P}$. We set $\rho_j := s_{2j-1}$, $\sigma_k := s_{2k}$, $1 \le j,k \le q$.

Firstly, we define matrices $\mathscr{C}(z)$ and $\mathscr{C}^{\#}(z)$ using formulae (4.1.6), (4.1.9), (4.1.10). We assume moreover the following open properties,

$$Q(z) := \det \mathscr{C}^{\#}(z) \neq 0\ ,\ \ z \in \overline{\mathbb{D}}\ ,\ \ \deg(Q) = N := q + \sum_{r=1}^{n} d_r\ . \tag{4.1.11}$$

We then define $\mathcal{H}(z) = (h_j(z))_{1\le j\le q}$ and $\mathcal{U}'(z) = (u_k'(z))_{1\le k\le q}$ to be the solutions of (4.1.8) and (4.1.5) respectively. We then define $\tau_j, \kappa_k > 0$ by

$$\prod_{j=1}^{q} \frac{1-x\sigma_j^2}{1-x\rho_j^2} = 1 + x\sum_{j=1}^{q} \frac{\tau_j^2}{1-x\rho_j^2} \tag{4.1.12}$$

$$\prod_{j=1}^{q} \frac{1-x\rho_j^2}{1-x\sigma_j^2} = 1 - x\left(\sum_{k=1}^{q} \frac{\kappa_k^2}{1-x\sigma_k^2}\right) \tag{4.1.13}$$

As a consequence, we recall from section 3.2 that the matrix

$$\mathcal{A} := \left(\frac{1}{\rho_j^2-\sigma_k^2}\right)_{1\le j,k\le q}$$

is invertible, with

$$\mathcal{A}^{-1} = \mathrm{diag}(\kappa_k^2)^t\mathcal{A}\mathrm{diag}(\tau_j^2)\ .$$

Similarly, the matrix

$$\mathcal{B} := \mathcal{A}\mathrm{diag}(\kappa_k^2)$$

satisfies

$$\mathcal{B}^{-1} =^t \mathcal{A}\mathrm{diag}(\tau_j^2)\ .$$

From these identities and

$$\mathscr{C}(z) = \mathrm{diag}(\rho_j)\mathcal{A} - z\mathrm{diag}(\Psi_{2j-1}(z))\mathcal{A}\mathrm{diag}(\sigma_k\Psi_{2k}(z)),$$

we easily prove the following lemma.

Lemma 10. — *For every $z\in\overline{\mathbb{D}}$,*

$$\mathscr{C}(z)\ {}^t\mathcal{B}\ \mathrm{diag}(\Psi_{2\ell-1}(z))_{1\le\ell\le q} = \mathrm{diag}(\Psi_{2j-1}(z))_{1\le j\le q}\ \mathcal{B}\ {}^t\mathscr{C}(z)\ .$$

In view of Lemma 10 and of the definition of $\mathcal{H}(z)$, we infer

$$\mathscr{C}(z)^t\mathcal{B}\mathrm{diag}(\Psi_{2j-1}(z))\mathcal{H}(z) = \mathrm{diag}(\Psi_{2j-1}(z))_{1\le j\le q}\ \mathcal{B}(\mathbf{1}) = \mathrm{diag}(\Psi_{2j-1}(z))_{1\le j\le q}$$

where we have used $\mathcal{B}(\mathbf{1}) = \mathbf{1}$, which is a consequence of identity (4.1.13). We conclude, in view of the definition of $\mathcal{U}'(z)$,

$$\mathcal{U}'(z) =^t \mathcal{B}\mathrm{diag}(\Psi_{2j-1}(z))\mathcal{H}(z), \tag{4.1.14}$$

or

$$u_k' = \kappa_k^2\sum_{j=1}^{q}\frac{\Psi_{2j-1}h_j}{\rho_j^2-\sigma_k^2}\ ,\ \Psi_{2j-1}h_j = \tau_j^2\sum_{k=1}^{q}\frac{u_k'}{\rho_j^2-\sigma_k^2}\ . \tag{4.1.15}$$

We are going to define antilinear operators H eand K on the space

$$W = \frac{\mathbb{C}_{N-1}[z]}{Q(z)}.$$

Cramers' formulae show that

$$h_j(z) = D_{2j-1}(z)R_{2j-1}(z)\ , \tag{4.1.16}$$

where the numerator of R_{2j-1} is a polynomial of degree at most $N-1-d_{2j-1}$, the denominator being Q. Similarly,

$$u'_k(z) = D_{2k}(z)R_{2k}(z)\ ,$$

where the numerator of R_{2k} is a polynomial of degree at most $N-1-d_{2k}$, the denominator being Q.

We can therefore define the following elements of W,

$$\begin{aligned} e_{2j-1,a}(z) &:= \frac{z^a}{D_{2j-1}(z)}h_j(z),\ 0\le a\le d_{2j-1}\ ,\\ e_{2k,b}(z) &:= \frac{z^b}{D_{2k}(z)}u'_k(z)\ ,\ 1\le b\le d_{2k}\ ,\end{aligned}$$

for $1\le j,k\le q$. Let

$$\begin{aligned} \mathcal{E} &:= \left((e_{2j-1,a})_{0\le a\le d_{2j-1}}\ ,\ (e_{2k,b})_{1\le b\le d_{2k}}\right)_{1\le j,k\le q}\ ,\\ \mathcal{E}' &:= \left((e_{2j-1,a})_{1\le a\le d_{2j-1}}\ ,\ (e_{2k,b})_{0\le b\le d_{2k}}\right)_{1\le j,k\le q}\ .\end{aligned}$$

We need a second open assumption.

$$\mathcal{E} \text{ and } \mathcal{E}' \text{ are bases of } W. \tag{4.1.17}$$

We then define antilinear operators H and K on W by

$$\begin{aligned} H(e_{2j-1,a}) &:= \rho_j \mathrm{e}^{-i\psi_{2j-1}} e_{2j-1,d_{2j-1}-a}\ ,\ 0\le a\le d_{2j-1}\ ,\\ H(e_{2k,b}) &:= \sigma_k \mathrm{e}^{-i\psi_{2k}} e_{2k,d_{2k}+1-b},\ 1\le b\le d_{2k}\ ,\\ K(e_{2j-1,a}) &:= \rho_j \mathrm{e}^{-i\psi_{2j-1}} e_{2j-1,d_{2j-1}-a-1}\ 0\le a\le d_{2j-1}-1\ ,\\ K(e_{2k,b}) &:= \sigma_k \mathrm{e}^{-i\psi_{2k}} e_{2k,d_{2k}-b},\ 0\le b\le d_{2k}\ .\end{aligned}$$

for $1\le j,k\le q$.

Notice that S^* acts on W. Indeed, if

$$Q(z) := Q(0)(1-c_1z-c_2z^2-\cdots-c_Nz^N),$$

we have

$$S^*\left(\frac{1}{Q}\right) = \sum_{j=1}^{N} c_j \frac{z^{j-1}}{Q(z)} \ .$$

As a first step, we prove that $S^*H = K$. Indeed, we just have to check this identity on each vector of the basis $\mathcal{E}$. In view of the above definition of H and K, the identity is trivial on the vectors

$$e_{2j-1,a} \ , \ 0 \le a \le d_{2j-1} - 1 \ , \ e_{2k,b} \ , \ 1 \le b \le d_{2k} \ .$$

It remains to prove it for $e_{2j-1,d_{2j-1}}$, or equivalently for $\Psi_{2j-1}h_j$. We have

$$S^*H(\Psi_{2j-1}h_j) = \rho_j S^* h_j \ .$$

This quantity can be calculated by applying S^* to the equation satisfied by $\mathcal{H}(z)$, namely

$${}^t\mathcal{A}\mathrm{diag}(\rho_j)\mathcal{H}(z) - z\mathrm{diag}(\sigma_k\Psi_{2k}(z))^t\mathcal{A}\mathrm{diag}\Psi_{2j-1}(z)\mathcal{H}(z) = \mathbf{1} \ .$$

We obtain

$$\begin{aligned} {}^t\mathcal{A}\mathrm{diag}(\rho_j)S^*\mathcal{H}(z) &= \mathrm{diag}(\sigma_k\Psi_{2k}(z))^t\mathcal{A}\mathrm{diag}\Psi_{2j-1}(z)\mathcal{H}(z) \\ &= \mathrm{diag}\left(\frac{\sigma_k\Psi_{2k}(z)}{\kappa_k^2}\right)\mathcal{U}'(z) \ , \end{aligned}$$

where we have used (4.1.14). We obtain, by using the formula for $\mathcal{A}^{-1}$,

$$S^*H(\Psi_{2j-1}h_j)(z) = \rho_j S^*h_j(z) = \tau_j^2 \sum_{k=1}^{q} \frac{\sigma_k\Psi_{2k}u_k'}{\rho_j^2 - \sigma_k^2} \ . \tag{4.1.18}$$

On the other hand, using the second part of (4.1.15),

$$K(\Psi_{2j-1}h_j) = \tau_j^2 \sum_{k=1}^{q} \frac{Ku_k'}{\rho_j^2 - \sigma_k^2} = \tau_j^2 \sum_{k=1}^{q} \frac{\sigma_k\Psi_{2k}u_k'}{\rho_j^2 - \sigma_k^2} \ .$$

Therefore we have proved $S^*H(\Psi_{2j-1}h_j) = K(\Psi_{2j-1}h_j)$, and finally $S^*H = K$.

As a next step, we show the following identity

$$\forall h \in W \ , \ S^*HS^*(h) = H(h) - (1|h)u \ , \ u := \sum_{k=1}^{q} u_k' \ . \tag{4.1.19}$$

It is enough to check this identity on any vector of the basis $\mathcal{E}$. In view of the definition of H, it is trivially satisfied on the vectors

$$e_{2j-1,a}\ ,\ 1\le a\le d_{2j-1}\ ,\ e_{2k,b}\ ,\ 1\le b\le d_{2k}\ .$$

Therefore we just have to check it on $e_{2j-1,0}$, or equivalently on h_j. On the one hand, we have, from the identity (4.1.18),

$$S^*h_j(z)=\frac{\tau_j^2}{\rho_j}\sum_{k=1}^q\frac{\sigma_k\Psi_{2k}u'_k}{\rho_j^2-\sigma_k^2}\ .$$

Applying $S^*H=K$, we get

$$\begin{aligned} S^*HS^*h_j(z) &= \frac{\tau_j^2}{\rho_j}\sum_{k=1}^q\frac{\sigma_k^2u'_k}{\rho_j^2-\sigma_k^2}\\ &= \rho_j\Psi_{2j-1}(z)h_j(z)-\frac{\tau_j^2}{\rho_j}u(z)\ , \end{aligned}$$

where we have used (4.1.15) again. We conclude by observing that $\rho_j\Psi_{2j-1}h_j=H(h_j)$, and that

$$(1|h_j)=\overline{h_j(0)}=\frac{\tau_j^2}{\rho_j}\ ,$$

in view of the equation on $\mathcal{H}(z)$ for $z=0$,

$$\mathcal{A}\mathrm{diag}(\rho_j)\mathcal{H}(0)=\mathbf{1}\ ,$$

and of the expression of $\mathcal{A}^{-1}=\mathrm{diag}(\tau_j^2)\mathcal{B}$.

Finally, we prove that an operator satisfying equality (4.1.19) is actually a Hankel operator.

Lemma 11. — *Let N be a positive integer. Let*

$$Q(z):=1-c_1z-c_2z^2-\dots-c_Nz^N$$

be a complex valued polynomial with no roots in the closed unit disc. Set

$$W:=\frac{\mathbb{C}_{N-1}[z]}{Q(z)}\subset L^2_+(\mathbb{S}^1)\ .$$

Let H be an antilinear operator on W satisfying

$$S^*HS^*=H-(1|\cdot)u$$

on W, for some $u \in W$. Then H coincides with the Hankel operator of symbol u on W.

Proof. — Consider the operator $\tilde{H} := H - H_u$, then $S^*\tilde{H}S^* = \tilde{H}$ on W and hence, it suffices to show that, if H is an antilinear operator on W such that $S^*HS^* = H$, then $H = 0$.

The family $(e_j)_{1\leq j\leq N}$ where

$$e_0(z) = \frac{1}{Q(z)}\ ,\ e_j(z) = S^j e_0(z),\ j = 1, \dots, N-1$$

is a basis of W. Using that

$$S^*HS^* = H$$

we get on the one hand that $He_k = (S^*)^k He_0$. On the other hand, since

$$S^*e_0 = S^*\left(\frac{1}{Q}\right) = \sum_{j=1}^{N} c_j e_{j-1}\ ,$$

this implies

$$He_0 = S^*HS^*e_0 = \sum_{j=1}^{N} \overline{c_j}(S^*)^j He_0\ ,$$

hence $\overline{Q}(S^*)H(e_0) = 0$. Observe that, by the spectral mapping theorem, the spectrum of $\overline{Q}(S^*)$ is contained into $\overline{Q}(\overline{\mathbb{D}})$, hence $\overline{Q}(S^*)$ is one-to-one. We conclude that $H(e_0) = 0$, and finally that $H = 0$.

□

Applying Lemma 11 to our vector space W, we conclude that $H = H_u$, and consequently $K = S^*H_u = K_u$. In view of the definition of H and K, we conclude that $\Phi_{d_1,\dots,d_n}(u) = \mathcal{P}$.

4.1.3.2. The mapping $\Phi_{d_1,\dots,d_n}$ is open from $\mathcal{V}_{(d_1,\dots,d_n)}$ to $\mathcal{S}_{(d_1,\dots,d_n)}$. — Notice that we have not yet completed the proof of Theorem 5 since the previous calculations were made under the assumptions (4.1.11) and (4.1.17). In other words, we proved that an element $\mathcal{P}$ of the target space satisfying (4.1.11) and (4.1.17) is in the range of $\Phi_{d_1,\dots,d_n}$. On the other hand, in section 4.1.2, we proved that these properties are satisfied by

the elements of the range of $\Phi_{d_1,\dots,d_n}$. Since these hypotheses are clearly open in the target space, we infer that the range of $\Phi_{d_1,\dots,d_n}$ is open.

4.1.3.3. The mapping $\Phi_{d_1,\dots,d_n}$ is closed.— Let (u^ε) be a sequence of $\mathcal{V}_{(d_1,\dots,d_n)}$ such that $\Phi_{d_1,\dots,d_n}(u^\varepsilon) := \mathcal{P}^\varepsilon$ converges to some $\mathcal{P}$ in $\mathcal{S}_{(d_1,\dots,d_n)}$ as ε goes to 0. In other words,

$$\mathcal{P}^\varepsilon = ((s_r^\varepsilon)_{1\le r\le 2q}, (\Psi_r^\varepsilon)_{1\le r\le 2q}) \longrightarrow \mathcal{P} = ((s_r)_{1\le r\le 2q}, (\Psi_r)_{1\le r\le 2q})$$

in $\mathcal{S}_{(d_1,\dots,d_n)}$ as $\varepsilon \to 0$. We have to find u such that $\Phi(u) = \mathcal{P}$. Since

$$\|u^\varepsilon\|_{H^{1/2}} \simeq \mathrm{Tr}(H_{u^\varepsilon}^2) = \sum_{r=1}^{2q} d_r (s_r^\varepsilon)^2 + \sum_{j=1}^{q} (s_{2j-1}^\varepsilon)^2$$

is bounded, we may assume, up to extracting a subsequence, that u^ε is weakly convergent to some u in $H_+^{1/2}$, and strongly convergent in L_+^2 by the Rellich theorem. Moreover, the rank of H_u is at most $N = q + \sum_{r=1}^{2q} d_r$.

Denote by u_j^ε the orthogonal projection of u^ε onto $\ker(H_u^2 - (s_{2j-1}^\varepsilon)^2 I)$, $j = 1,\dots,q$, and by $(u_k^\varepsilon)'$ the orthogonal projection of u^ε onto $\ker(K_u^2 - (s_{2k}^\varepsilon)^2 I)$, $k = 1,\dots,q$. Since all these functions are bounded in L_+^2, we may assume that, for the weak convergence in L_+^2,

$$u_j^\varepsilon \rightharpoonup v_j \ , \ (u_k^\varepsilon)' \rightharpoonup v_k' \ .$$

Taking advantage of the strong convergence of u^ε in L_+^2, we can pass to the limit in

$$(u^\varepsilon|u_j^\varepsilon) = \|u_j^\varepsilon\|^2 =: (\tau_j^\varepsilon)^2 \ , \ (u^\varepsilon|(u_k^\varepsilon)') = \|(u_k^\varepsilon)'\|^2 =: (\kappa_k^\varepsilon)^2 \ ,$$

and obtain, thanks to the explicit expressions (3.2.5), (3.2.6) of τ_j^2, κ_k^2 in terms of the s_r,

$$(u|v_j) = \tau_j^2 > 0 \ , \ (u|v_k') = \kappa_k^2 > 0 \ ,$$

in particular $v_j \ne 0, v_k' \ne 0$ for every j, k.

On the other hand, passing to the limit in

$$\begin{aligned} s_{2j-1}^\varepsilon u_j^\varepsilon &= \Psi_{2j}^\varepsilon \ H_{u^\varepsilon} u_j^\varepsilon \ , \ H_{u^\varepsilon}^2(u_j^\varepsilon) = (s_{2j-1}^\varepsilon)^2 u_j^\varepsilon \ , \\ K_{u^\varepsilon}(u_k^\varepsilon)' &= s_{2k}^\varepsilon \Psi_{2k}^\varepsilon \ (u_k^\varepsilon)' \ , \ K_{u^\varepsilon}^2(u_k^\varepsilon)' = (s_{2k}^\varepsilon)^2 (u_k^\varepsilon)' \ , \\ u^\varepsilon &= \sum_{j=1}^q u_j^\varepsilon = \sum_{k=1}^q (u_k^\varepsilon)' \ , \end{aligned}$$

we obtain

$$\begin{aligned} s_{2j-1}v_j &= \Psi_{2j}\; H_u v_j\ ,\ H_u^2(v_j) = s_{2j-1}^2 v_j\ ,\\ K_u v'_k &= s_{2k}\Psi_{2k}\; v'_k\ ,\ K_u^2(v'_k) = s_{2k}^2 v'_k\ ,\\ u &= \sum_{j=1}^q v_j = \sum_{k=1}^q v'_k\ . \end{aligned}$$

This implies that $u \in \mathcal{V}_{(d_1,\dots,d_n)}$, $v_j = u_j, v'_k = u'_k$, and $\Phi(u) = \mathcal{P}$. The proof of Theorem 5 is thus complete in the case $n = 2q$, under the assumption that $\mathcal{V}_{(d_1,\dots,d_n)}$ is non empty.

4.1.4. $\mathcal{V}_{(d_1,\dots,d_n)}$ is non empty, n even. — Let n be a positive even integer. The aim of this section is to prove that $\mathcal{V}_{(d_1,\dots,d_n)}$ is not empty for any multi-index $(d_1,\dots,d_n)$ of non negative integers.

The preceding section implies that, as soon as $\mathcal{V}_{(d_1,\dots,d_n)}$ is non empty, it is homeomorphic to $\mathcal{S}_{(d_1,\dots,d_n)}$, via the explicit formula (1.0.5). We argue by induction on the integer $d_1 + \cdots + d_n$. In the generic case consisting of simple eigenvalues (see [11]), we proved that for any positive integer q, $\mathcal{V}_{(0,\dots,0)}$ (which was denoted by $\mathcal{V}_{\rm gen}(2q)$ in [11]) is non empty. As a consequence, to any given sequence $((s_r),(\Psi_r)) \in \Omega_{2q} \times \mathbb{T}^{2q}$ corresponds a unique $u \in \mathcal{V}_{(0,\dots,0)}$, the s_{2j-1}^2 being the simple eigenvalues of H_u^2 and the s_{2j}^2 the simple eigenvalues of K_u^2. This gives the theorem in the case $(d_1,\dots,d_n) = (0,\dots,0)$ for every n, which is one of the main theorems of [12]. Let us turn to the induction argument, which is clearly a consequence of the following lemma.

Lemma 12. — *Let $n = 2q$, $(d_1,\dots,d_n)$ and $1 \le r \le n-1$. Assume*

$$\mathcal{V}_{(d_1,\dots,d_r,0,0,d_{r+1},\dots,d_n)} \textit{ is non empty,}$$

then

$$\mathcal{V}_{(d_1,\dots,d_{r-1},d_r+1,d_{r+1},\dots,d_n)} \textit{ is non empty.}$$

Proof. — The main idea is to collapse three consecutive singular values. More precisely, we will construct elements of $\mathcal{V}_{(d_1,\dots,d_{r-1},d_r+1,d_{r+1},\dots,d_n)}$ as limits of sequences in $\mathcal{V}_{(d_1,\dots,d_r,0,0,d_{r+1},\dots,d_n)}$ such that s_r, s_{r+1}, s_{r+2} converge to the same value. The difficulty is to make the sequence converge

strongly, and we will see that this requires some non resonance assumption on the corresponding angles.

We consider the case $r = 1$. The proof in the other cases follows the same lines. Write $m_j := d_{2j-1} + 1$ and $\ell_k := d_{2k} + 1$. From the assumption,

$$\mathcal{V} := \mathcal{V}_{(d_1,0,0,d_2,\dots,d_n)} \text{ is non empty,}$$

hence Φ establishes a diffeomorphism from $\mathcal{V}$ into

$$\mathcal{S}_{(d_1,0,0,d_2,\dots,d_n)} \ .$$

Therefore, given $\rho > \sigma_2 > \rho_3 > \sigma_3 > \cdots > \rho_{q+1} > \sigma_{q+1} > 0$, and $\Psi_1, \theta_1, \varphi_2, \Psi_4, \dots, \Psi_{n+2}$, for every $\varepsilon > 0$ small enough, we define

$$u^\varepsilon := u((s_1^\varepsilon, \dots, s_{2(q+1)}^\varepsilon), (\Psi_1, \dots, \Psi_{2(q+1)}))$$

with

$$\begin{aligned} s_1^\varepsilon &:= \rho + \varepsilon \ , \ s_2^\varepsilon := \rho \ , \ s_3^\varepsilon := \rho - \varepsilon \ , \ s_{2k}^\varepsilon := \sigma_k \ , \ 2 \le k \le q+1 \ , \\ s_{2j-1}^\varepsilon &:= \rho_j \ , \ 3 \le j \le q+1 \ , \Psi_2 := \mathrm{e}^{-i\theta_1} \ , \ \Psi_3 := \mathrm{e}^{-i\varphi_2} \ . \end{aligned}$$

By making ε go to 0, we are going to construct u in $\mathcal{V}_{(d_1+1,d_2,\dots,d_n)}$, such that $s_1(u) = \rho$ is of multiplicity $m_1 + 1 = d_1 + 2$, $s_{2j-1}(u) = \rho_{j+1}$, $j = 2, \dots, q$, is of multiplicity m_j and $s_{2k}(u) = \sigma_{k+1}$ for $k = 1, \dots, q$, is of multiplicity ℓ_k.

First of all, observe that u^ε is bounded in $H^{1/2}_+$, since its norm is equivalent to

$$\mathrm{Tr}(H^2_{u^\varepsilon}) = (d_1 + 1)(\rho + \varepsilon)^2 + (\rho - \varepsilon)^2 + d_2\sigma_2^2 + (d_3 + 1)\rho_3^2 + \dots$$

Hence, by the Rellich theorem, up to extracting a subsequence, u^ε strongly converges in L^2_+ to some $u \in H^{1/2}_+$. Similarly, the orthogonal projections u_j^ε and $(u_k^\varepsilon)'$ are bounded in L^2_+, hence are weakly convergent

to v_j, v'_k. Arguing as in the previous subsection, we have

$$\begin{aligned}
(u|v_1) &= \lim_{\varepsilon\to 0}\|u_1^\varepsilon\|^2 = \lim_{\varepsilon\to 0}\frac{(\rho+\varepsilon)^2-\rho^2}{(\rho+\varepsilon)^2-(\rho-\varepsilon)^2}\frac{\prod_{k\geq 2}((\rho+\varepsilon)^2-\sigma_k^2)}{\prod_{k\geq 3}((\rho+\varepsilon)^2-\rho_k^2)}\\
&= \frac{1}{2}\frac{\prod_{k\geq 2}(\rho^2-\sigma_k^2)}{\prod_{k\geq 3}(\rho^2-\rho_k^2)},\\
(u|v_2) &= \lim_{\varepsilon\to 0}\|u_2^\varepsilon\|^2 = \lim_{\varepsilon\to 0}\frac{(\rho-\varepsilon)^2-\rho^2}{(\rho-\varepsilon)^2-(\rho+\varepsilon)^2}\frac{\prod_{k\geq 2}((\rho-\varepsilon)^2-\sigma_k^2)}{\prod_{k\geq 3}((\rho-\varepsilon)^2-\rho_k^2)}\\
&= \frac{1}{2}\frac{\prod_{k\geq 2}(\rho^2-\sigma_k^2)}{\prod_{k\geq 3}(\rho^2-\rho_k^2)},
\end{aligned}$$

$$\begin{aligned}
(u|v_j) &= \lim_{\varepsilon\to 0}\|u_j^\varepsilon\|^2\ , j\geq 3,\\
&= \lim_{\varepsilon\to 0}\frac{\rho_j^2-\rho^2}{(\rho_j^2-(\rho-\varepsilon)^2)(\rho_j^2-(\rho+\varepsilon)^2)}\frac{\prod_{k\geq 2}(\rho_j^2-\sigma_k^2)}{\prod_{k\geq 3,k\neq j}(\rho_j^2-\rho_k^2)}\\
&= \frac{1}{\rho_j^2-\rho^2}\frac{\prod_{k\geq 2}(\rho_j^2-\sigma_k^2)}{\prod_{k\geq 3,k\neq j}(\rho_j^2-\rho_k^2)}
\end{aligned}$$

and

$$\begin{aligned}
(u|v'_1) &= \lim_{\varepsilon\to 0}\|(u_1^\varepsilon)'\|^2\\
&= \lim_{\varepsilon\to 0}(\rho^2-(\rho+\varepsilon)^2)(\rho^2-(\rho-\varepsilon)^2)\frac{\prod_{k\geq 3}(\rho^2-\rho_k^2)}{\prod_{k\geq 2}(\rho^2-\sigma_k^2)} = 0,\\
(u|v'_k) &= \lim_{\varepsilon\to 0}\|(u_k^\varepsilon)'\|^2\ ,\ k\geq 2\\
&= \lim_{\varepsilon\to 0}\frac{(\sigma_k^2-(\rho+\varepsilon)^2)(\sigma_k^2-(\rho-\varepsilon)^2)}{\sigma_k^2-\rho^2}\frac{\prod_{j\geq 3}(\sigma_k^2-\rho_j^2)}{\prod_{j\geq 2,j\neq k}(\sigma_k^2-\sigma_j^2)}\\
&= \frac{1}{\sigma_k^2-\rho^2}\frac{\prod_{j\geq 3}(\sigma_k^2-\rho_j^2)}{\prod_{j\geq 2,j\neq k}(\sigma_k^2-\sigma_j^2)}
\end{aligned}$$

In view of these identities, we infer that $v_j, j \geq 1$ and $v'_k, k \geq 2$ are not 0, while $v'_1 = 0$. Passing to the limit into the identities

$$\begin{aligned} s^\varepsilon_{2j-1} u^\varepsilon_j &= \Psi_{2j}\ H_{u^\varepsilon} u^\varepsilon_j\ ,\ H^2_{u^\varepsilon}(u^\varepsilon_j) = (s^\varepsilon_{2j-1})^2 u^\varepsilon_j\ , \\ K_{u^\varepsilon}(u^\varepsilon_k)' &= s^\varepsilon_{2k} \Psi_{2k}\ (u^\varepsilon_k)'\ ,\ K^2_{u^\varepsilon}(u^\varepsilon_k)' = (s^\varepsilon_{2k})^2 (u^\varepsilon_k)'\ , \\ u^\varepsilon &= \sum_{j=1}^{q} u^\varepsilon_j = \sum_{k=1}^{q} (u^\varepsilon_k)'\ , \end{aligned}$$

we obtain

$$\begin{aligned} s_{2j-1} v_j &= \Psi_{2j}\ H_u v_j\ ,\ H^2_u(v_j) = s^2_{2j-1} v_j\ , \\ K_u v'_k &= s_{2k} \Psi_{2k}\ v'_k\ ,\ K^2_u(v'_k) = s^2_{2k} v'_k\ , \\ u &= \sum_{j=1}^{q+1} v_j = \sum_{k=2}^{q+1} v'_k\ , \end{aligned}$$

hence

$$\dim E_u(\rho_j) \geq m_j = d_{2j-1}+1\ ,\ j \geq 3,\ \dim F_u(\sigma_k) \geq \ell_k = d_{2k}+1\ ,\ k \geq 2\ .$$

On the other hand, we know that

$$\begin{aligned} \mathrm{rk}(H_{u^\varepsilon}) + \mathrm{rk}(K_{u^\varepsilon}) &= (2d_1+1)+1+1+(2d_2+1)+\cdots+(2d_{2q}+1) \\ &= [2(d_1+1)+1] + \sum_{r=2}^{2q} (2d_r+1), \end{aligned}$$

and, consequently, $\mathrm{rk}(H_u) + \mathrm{rk}(K_u)$ is not bigger than the right hand side. In order to conclude that $u \in \mathcal{V}_{(d_1+1, d_2, \dots, d_n)}$, it therefore remains to prove that

$$\dim E_u(\rho) \geq m_1 + 1 = d_1 + 2\ .$$

We use the explicit formulae obtained in section 4.1.2, which read

$$u^\varepsilon_j(z) = \Psi_{2j-1}(z) \frac{\det \mathscr{C}^\varepsilon_j(z)}{\det \mathscr{C}^\varepsilon(z)}\ ,$$

where $\mathscr{C}^\varepsilon(z)$ denotes the matrix $(c^\varepsilon_{k\ell}(z))_{1\leq k,\ell \leq q+1}$, with

$$c^\varepsilon_{jk}(z) = \frac{\rho_j - \sigma_k z \Psi_{2j-1}(z) \Psi_{2k}(z)}{\rho_j^2 - \sigma_k^2}\ ,\ \rho_1 = \rho+\varepsilon, \sigma_1 = \rho, \rho_2 = \rho - \varepsilon\ ,$$

and $\mathscr{C}_j^\varepsilon(z)$ denotes the matrix deduced from $\mathscr{C}^\varepsilon(z)$ by replacing the line j by the line $(1,\dots,1)$. Notice that elements $c_{11}^\varepsilon(z)$ and $c_{21}^\varepsilon(z)$ in formulae (4.1.6) are of order ε^{-1}, hence we compute

$$\lim_{\varepsilon\to 0} 2\varepsilon \det \mathscr{C}^\varepsilon(z) = \begin{vmatrix} 1 - z\mathrm{e}^{-i\theta_1}\Psi_1 & \dfrac{\rho - z\sigma_2\Psi_1\Psi_4}{\rho^2-\sigma_2^2} & \dots \\ -(1 - z\mathrm{e}^{-i(\theta_1+\varphi_2)}) & \dfrac{\rho - z\sigma_2\mathrm{e}^{-i\varphi_2}\Psi_4}{\rho^2-\sigma_2^2} & \dots \\ 0 & \dots & \dots \end{vmatrix} .$$

We compute this determinant by using the following formula

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ 0 & a_{32} & \dots & a_{3N} \\ 0 & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots \end{vmatrix} = \begin{vmatrix} b_2 & \dots & b_N \\ a_{32} & \dots & a_{3N} \\ \dots & \dots & \dots \\ \vdots & \vdots & \vdots \\ \dots & \dots & \dots \end{vmatrix} , \; b_k := a_{11}a_{2k} - a_{21}a_{1k} \, , \; k \geq 2 \, .$$

Observe that

$$\begin{aligned} b_k &= (1 - z\mathrm{e}^{-i\theta_1}\Psi_1)\frac{\rho - z\sigma_k\mathrm{e}^{-i\varphi_2}\Psi_{2k}}{\rho^2-\sigma_k^2} + (1 - z\mathrm{e}^{-i(\theta_1+\varphi_2)})\frac{\rho - z\sigma_k\Psi_1\Psi_{2k}}{\rho^2-\sigma_k^2} \\ &= 2(1-q(z)z)\left(\frac{\rho - z\sigma_k\tilde{\Psi}_1\Psi_{2k}}{\rho^2-\sigma_k^2}\right) \end{aligned}$$

where

$$q(z) = \mathrm{e}^{-i\theta_1}\frac{\Psi_1(z) + \mathrm{e}^{-i\varphi_2}}{2}$$

and

$$\tilde{\Psi}_1(z) = \frac{z\mathrm{e}^{-i(\pi+\theta_1+\varphi_2)}\Psi_1(z) + \frac{\Psi_1(z)+\mathrm{e}^{-i\varphi_2}}{2}}{1-q(z)z}.$$

We obtain

$$\lim_{\varepsilon\to 0} 2\varepsilon \det \mathscr{C}^\varepsilon(z) = 2(1-q(z)z)\det((\tilde{c}_{jk})_{2\leq j,k\leq q+1})$$

where, for $k \geq 2$, $j \geq 3$,

$$\begin{aligned} \tilde{c}_{2k} &= \frac{\rho - z\sigma_k\tilde{\Psi}_1(z)\Psi_{2k}(z)}{(\rho^2-\sigma_k^2)} \\ \tilde{c}_{jk} &= c_{jk} = \frac{\rho_j - z\sigma_k\Psi_{2j-1}(z)\Psi_{2k}(z)}{\rho_j^2-\sigma_k^2} \, . \end{aligned}$$

We know that Ψ_1 is a Blaschke product of degree $m_1 - 1$. Let us verify that it is possible to choose φ_2 so that $\tilde{\Psi}_1$ is a Blaschke product of degree m_1. We first claim that it is possible to choose φ_2 so that $1 - q(z)z \neq 0$ for $|z| \leq 1$. Assume $1 - q(z)z = 0$. Then

$$1 = \frac{1}{2}\left(\mathrm{e}^{-i\theta_1}\Psi_1(z)z + \mathrm{e}^{-i(\varphi_2+\theta_1)}z\right) . \tag{4.1.20}$$

First notice that this clearly imposes $|z| = 1$. Furthermore, this implies equality in the Minkowski inequality, therefore there exists $\lambda > 0$ so that $\Psi_1(z) = \lambda \mathrm{e}^{-i\varphi_2}$ and, eventually, that $\Psi_1(z) = \mathrm{e}^{-i\varphi_2}$ since $|\Psi_1(z)| = 1$. Inserting this in equation (4.1.20) gives $z = \mathrm{e}^{i(\varphi_2+\theta_1)}$ so that $\Psi_1(\mathrm{e}^{i(\varphi_2+\theta_1)}) = \mathrm{e}^{-i\varphi_2}$. If this equality holds true for any choice of φ_2, by analytic continuation inside the unit disc, we would have

$$\Psi_1(z) = \frac{\mathrm{e}^{i\theta_1}}{z}$$

which is not possible since Ψ_1 is a holomorphic function in the unit disc. Hence, one can choose φ_2 in order to have $1 - q(z)z \neq 0$ for any $|z| \leq 1$. It implies that $\tilde{\Psi}_1$ is a holomorphic rational function in the unit disc. Moreover, if $|z| = 1$,

$$\tilde{\Psi}_1(z) = \mathrm{e}^{-i(\pi+\theta_1+\varphi_2)}\Psi_1(z)\frac{z - \overline{q(z)}}{1 - zq(z)} ,$$

hence $|\tilde{\Psi}_1(z)| = 1$.

We conclude that $\tilde{\Psi}_1$ is a Blaschke product. Finally, its degree is $\deg(\Psi_1) + 1 = m_1$.

Next, we perform the same calculation with the numerator of $\det \mathscr{C}_j^\varepsilon(z)$ for $j = 1, 2$. We compute

$$\begin{aligned}
&\lim_{\varepsilon\to 0} 2\varepsilon \det \mathscr{C}_1^\varepsilon(z) = \\
&= \begin{vmatrix} 0 & 1 & \ldots 1 \\ -(1 - z\mathrm{e}^{-i(\theta_1+\varphi_2)}) & \dfrac{\rho - z\sigma_2\mathrm{e}^{-i\varphi_2}\Psi_4}{\rho^2 - \sigma_2^2} & \ldots \\ 0 & \ldots & \ldots \end{vmatrix} \\
&= (1 - z\mathrm{e}^{-i(\theta_1+\varphi_2)}) \det \begin{pmatrix} 1 & \ldots & 1 \\ c_{j2} & \ldots & c_{j\,q+1} \end{pmatrix}_{j\geq 3}
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0} 2\varepsilon \det \mathscr{C}_2^\varepsilon(z) = \\
= &\begin{vmatrix} 1-z\mathrm{e}^{-i\theta_1}\Psi_1 & \dfrac{\rho - z\sigma_2\Psi_1\Psi_4}{\rho^2-\sigma_2^2} & \dots \\ 0 & 1 & \dots 1 \\ 0 & \dfrac{\rho_3 - z\sigma_2\Psi_5\Psi_4}{\rho_3^2-\sigma_2^2} & \dots \end{vmatrix} \\
= &(1-z\mathrm{e}^{-i\theta_1}\Psi_1)\det\begin{pmatrix} 1 & \dots & 1 \\ c_{j2} & \dots & c_{j\,q+1}\end{pmatrix}_{j\ge 3}
\end{aligned}$$

Hence we have, for the weak convergence in L^2_+,

$$\begin{aligned}
v_1(z) &:= \lim_{\varepsilon\to 0} u_1^\varepsilon(z) = \Psi_1(z)\frac{(1-z\mathrm{e}^{-i(\theta_1+\varphi_2)})}{2(1-q(z)z)}\cdot\frac{\det\begin{pmatrix} 1 & \dots & 1 \\ c_{j2} & \dots & c_{j\,q+1}\end{pmatrix}_{j\ge 3}}{\det\left((\tilde c_{jk})_{2\le j,k\le q+1}\right)} \\
v_2(z) &:= \lim_{\varepsilon\to 0} u_2^\varepsilon(z) = \mathrm{e}^{-i\varphi_2}\frac{(1-z\mathrm{e}^{-i\theta_1}\Psi_1(z))}{2(1-q(z)z)}\cdot\frac{\det\begin{pmatrix} 1 & \dots & 1 \\ c_{j2} & \dots & c_{j\,q+1}\end{pmatrix}_{j\ge 3}}{\det\left((\tilde c_{jk})_{2\le j,k\le q+1}\right)}\ .
\end{aligned}$$

Furthermore, if D_1 denotes the normalized denominator of Ψ_1, we have

$$\begin{aligned}
H^2_{u^\varepsilon}\left(\frac{z^a}{D_1(z)}\frac{u_1^\varepsilon}{\Psi_1}\right) &= (\rho+\varepsilon)^2\frac{z^a}{D_1(z)}\frac{u_1^\varepsilon}{\Psi_1},\ 0\le a\le m_1-1, \\
H^2_{u^\varepsilon}(u_2^\varepsilon) &= (\rho-\varepsilon)^2 u_2^\varepsilon,
\end{aligned}$$

Passing to the limit in these identities as ε goes to 0, we get

$$\begin{aligned}
H^2_u\left(\frac{z^a}{D_1(z)}\frac{v_1}{\Psi_1}\right) &= \rho^2\frac{z^a}{D_1(z)}\frac{v_1}{\Psi_1},\ 0\le a\le m_1-1, \\
H^2_u(v_2) &= \rho^2 v_2\ .
\end{aligned}$$

It remains to prove that the dimension of the vector space generated by

$$v_2\ ,\ \frac{z^a}{D_1(z)}\frac{v_1}{\Psi_1},\ 0\le a\le m_1-1,$$

is m_1+1. From the expressions of v_1 and v_2, it is equivalent to prove that the dimension of the vector space spanned by the functions

$$(1-\mathrm{e}^{-i\theta_1}z\Psi_1(z))\ ,\ \frac{z^a}{D_1(z)}(1-\mathrm{e}^{-i(\varphi_2+\theta_1)}z),\ 0\le a\le m_1-1,$$

is m_1+1. We claim that our choice of φ_2 implies that this family is free. Indeed, assume that for some coefficients λ_a, $0\le a\le m_1-1$, we have

$$\sum_{a=0}^{m_1-1}\lambda_a\frac{z^a}{D_1(z)}=\frac{1-\mathrm{e}^{-i\theta_1}z\Psi_1(z)}{1-\mathrm{e}^{-i(\varphi_2+\theta_1)}z}$$

then, as the left hand side is a holomorphic function in $\overline{\mathbb{D}}$, it would imply $\Psi_1(\mathrm{e}^{i(\varphi_2+\theta_1)})=\mathrm{e}^{-i\varphi_2}$ but φ_2 has been chosen so that this does not hold. This completes the proof. □

4.1.5. The case n odd.— The proof of the fact that $\Phi_{d_1,\dots,d_n}$ is one-to-one is the same as in the case n even. One has to prove that $\Phi_{d_1,\dots,d_n}$ is onto. We shall proceed by approximation from the case n even. We define $q=\frac{n+1}{2}$.

Let

$$\mathcal{P}=((\rho_1,\sigma_1,\dots,\rho_q),(\Psi_r)_{1\le r\le n})$$

be an arbitrary element of $\mathcal{S}_{(d_1,\dots,d_n)}$. We look for $u\in\mathcal{V}_{(d_1,\dots,d_n)}$ such that $\Phi_{d_1,\dots,d_n}(u)=\mathcal{P}$. Consider, for every ε such that $0<\varepsilon<\rho_q$,

$$\mathcal{P}_\varepsilon=((\rho_1,\sigma_1,\dots,\rho_q,\varepsilon),((\Psi_r)_{1\le r\le n},1))\in\mathcal{S}_{(d_1,\dots,d_n,0)}$$

– we take $\Psi_{2q}=1\in\mathcal{B}_0$. From Theorem 5, we get $u_\varepsilon\in\mathcal{V}_{(d_1,\dots,d_{n+1})}$ such that $\Phi(u_\varepsilon)=\mathcal{P}_\varepsilon$. As before, we can prove by a compactness argument that a subsequence of u_ε has a limit $u\in\mathcal{V}_{(d_1,\dots,d_n)}$ as ε tends to 0 with $\Phi_{d_1,\dots,d_n}(u)=\mathcal{P}$. We leave the details to the reader. □

4.1.6. $\mathcal{V}_{(d_1,\dots,d_n)}$ is a manifold. — Let $d=n+2\sum_r d_r$. We consider the map

$$\begin{array}{rcl}\Theta:\mathcal{S}_{(d_1,\dots,d_n)} & \longrightarrow & \mathcal{V}(d)\\ (\mathbf{s},\boldsymbol{\Psi}) & \longmapsto & u(\mathbf{s},\boldsymbol{\Psi})\end{array}$$

This map is well defined and C^∞ on $\mathcal{S}_{(d_1,\dots,d_n)}$. Moreover, from the previous section, it is a homeomorphism onto its range $\mathcal{V}_{(d_1,\dots,d_n)}$. In order to prove that $\mathcal{V}_{(d_1,\dots,d_n)}$ is a submanifold of $\mathcal{V}(d)$, it is enough to check that the differential of Θ is injective at every point. From Lemma 9, near

every point $u_0 \in \mathcal{V}_{(d_1,\dots,d_n)}$, there exists a smooth function $\tilde{\Phi}_n$, defined on a neighborhood V on u_0 in $\mathcal{V}(d)$, such that $\tilde{\Phi}_n$ coincides with $\Phi_{d_1,\dots,d_n}$ on $V \cap \mathcal{V}_{(d_1,\dots,d_n)}$. Consequently, $\tilde{\Phi}_n \circ \Theta$ is the identity on a neighborhood of $\mathcal{P}_0 := \Phi_{d_1,\dots,d_n}(u_0)$. In particular, the differential of Θ at $\mathcal{P}_0$ is injective. Finally, the dimension of $\mathcal{V}_{(d_1,\dots,d_n)}$ is

$$\dim \mathcal{S}_{d_1,\dots,d_n} = \dim \Omega_n + \sum_{r=1}^{n} \dim \mathcal{B}_{d_r} = 2n + 2\sum_{r=1}^{n} d_r,$$

using Proposition 2.

4.2. Extension to the Hilbert-Schmidt class

In this section, we consider infinite rank Hankel operators in the Hilbert-Schmidt class. We set

$$\mathcal{V}^{(2)}_{(d_r)_{r\geq 1}} := \Phi^{-1}(\mathcal{S}^{(2)}_{(d_r)_{r\geq 1}}).$$

Theorem 6. — *The mapping*

$$\begin{array}{rcl} \Phi : \mathcal{V}^{(2)}_{(d_r)_{r\geq 1}} & \longrightarrow & \mathcal{S}^{(2)}_{(d_r)_{r\geq 1}} \\ u & \longmapsto & ((s_r)_{r\geq 1}, (\Psi_r)_{r\geq 1}) \end{array}$$

is a homeomorphism. Furthermore, if $(s_r, \Psi_r)_{r\geq 1} \in \mathcal{S}^{(2)}_\infty$, then its preimage u by Φ is given by

$$u(z) = \lim_{q\to\infty} u_q(z) \tag{4.2.1}$$

where

$$u_q := u((s_1,\dots,s_{2q}),(\Psi_1,\dots,\Psi_{2q})).$$

Proof. — The fact that Φ is one-to-one follows from an explicit formula analogous to the one obtained in the finite rank case, see section 4.1.2. However, in this infinite rank situation, we have to proceed slightly differently, in order to deal with the continuity of infinite rank matrices on appropriate ℓ^2 spaces.

Indeed, we still have

$$u = \sum_{j=1}^{\infty} \Psi_{2j-1} h_j$$

where $\mathcal{H}(z) := (h_j(z))_{j\geq 1}$ satisfies the following infinite dimensional system, for every $z \in \mathbb{D}$,

$$\mathcal{H}(z) = \mathcal{F}(z) + z\mathcal{D}(z)\mathcal{H}(z) \tag{4.2.2}$$

with

$$\begin{aligned} \mathcal{F}(z) &:= \left(\frac{\tau_j^2}{\rho_j}\right)_{j\geq 1}, \\ \mathcal{D}(z) &:= \left(\frac{\tau_j^2}{\rho_j}\sum_{k=1}^{\infty}\frac{\kappa_k^2\sigma_k\Psi_{2k}(z)\Psi_{2\ell-1}(z)}{(\rho_j^2-\sigma_k^2)(\rho_\ell^2-\sigma_k^2)}\right)_{j,\ell\geq 1}. \end{aligned}$$

In order to derive this system, just write

$$\begin{aligned} h_j(z) &= \frac{1}{\rho_j}H_u(u_j)(z) = \frac{\tau_j^2}{\rho_j} + \frac{1}{\rho_j}SK_u(u_j)(z) \\ &= \frac{\tau_j^2}{\rho_j}\left(1 + z\sum_{k=1}^{\infty}\frac{K_u(u_k')(z)}{\rho_j^2-\sigma_k^2}\right) = \frac{\tau_j^2}{\rho_j}\left(1 + z\sum_{k=1}^{\infty}\frac{\sigma_k\Psi_{2k}(z)u_k'(z)}{\rho_j^2-\sigma_k^2}\right) \\ &= \frac{\tau_j^2}{\rho_j}\left(1 + z\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{\kappa_k^2\sigma_k\Psi_{2k}(z)\Psi_{2\ell-1}(z)h_\ell(z)}{(\rho_j^2-\sigma_k^2)(\rho_\ell^2-\sigma_k^2)}\right). \end{aligned}$$

Notice that the coefficients of the infinite matrix $\mathcal{D}(z)$ depend holomorphically on $z \in \mathbb{D}$. We are going to prove that, for every $z \in \mathbb{D}$, $\mathcal{D}(z)$ defines a contraction on the space ℓ^2_τ of sequences $(v_j)_{j\geq 1}$ satisfying

$$\sum_{j=1}^{\infty}\frac{|v_j|^2}{\tau_j^2} < \infty .$$

From the maximum principle, we may assume that z belongs to the unit circle. Then z and $\Psi_r(z)$ have modulus 1. We then compute $\mathcal{D}(z)\mathcal{D}(z)^*$, where the adjoint is taken for the inner product associated to ℓ^2_τ. We get,

using identities (3.2.12), (3.2.11) and (3.2.13),

$$\begin{aligned}[\mathcal{D}(z)\mathcal{D}(z)^*]_{jn} &= \frac{\tau_j^2}{\rho_j\rho_n}\sum_{k,\ell,m}\frac{\kappa_k^2\sigma_k\Psi_{2k}(z)\tau_\ell^2\kappa_m^2\sigma_m\overline{\Psi_{2m}(z)}}{(\rho_j^2-\sigma_k^2)(\rho_\ell^2-\sigma_k^2)(\rho_n^2-\sigma_m^2)(\rho_\ell^2-\sigma_m^2)}\\ &= \frac{\tau_j^2}{\rho_j\rho_n}\sum_{k}\frac{\kappa_k^2\sigma_k^2}{(\rho_j^2-\sigma_k^2)(\rho_n^2-\sigma_k^2)}\\ &= -\frac{\tau_j^2}{\rho_j\rho_n}+\delta_{jn}\ .\end{aligned}$$

Since, from the identity (3.2.3),

$$\sum_{j=1}^{\infty}\frac{\tau_j^2}{\rho_j^2}\le 1\ ,$$

we conclude that $\mathcal{D}(z)\mathcal{D}(z)^*\le I$ on ℓ^2_τ, and consequently that

$$\|\mathcal{D}(z)\|_{\ell^2_\tau\to\ell^2_\tau}\le 1\ .$$

From the Cauchy inequalities, this implies

$$\|\mathcal{D}^{(n)}(0)\|_{\ell^2_\tau\to\ell^2_\tau}\le n!\ .$$

Coming back to equation (4.2.2), we observe that $\mathcal{H}(0)=\mathcal{F}(0)\in\ell^2_\tau$, and that, for every $n\ge 0$,

$$\mathcal{H}^{(n+1)}(0)=(n+1)\sum_{p=0}^{n}\binom{n}{p}\mathcal{D}^{(p)}(0)\mathcal{H}^{(n-p)}(0)\ .$$

By induction on n, this determines $\mathcal{H}^{(n)}(0)\in\ell^2_\tau$, whence the injectivity of Φ.

Next, we prove that Φ is onto. We pick an element

$$\mathcal{P}\in\mathcal{S}^{(2)}_{(d_r)}$$

and we construct $u\in H^{1/2}_+$ so that $\Phi(u)=\mathcal{P}$. Set

$$\mathcal{P}=((\rho_1,\sigma_1,\rho_2,\dots),(\Psi_r)_{r\ge 1})$$

and consider, for any integer N,

$$\mathcal{P}_N:=((\rho_1,\sigma_1,\dots,\rho_N,\sigma_N),(\Psi_r)_{1\le r\le 2N})$$

in

$$\mathcal{S}_{(d_1,\dots,d_{2N})}\ .$$

From Theorem 5, there exists $u_N \in \mathcal{V}_{(d_1,\dots,d_{2N})}$ with $\Phi(u_N) = \mathcal{P}_N$. As

$$\mathrm{Tr}(H_{u_N}^2) = \sum_{r=1}^{2N} d_{2r} s_{2r}^2 + \sum_{j=1}^{N} (d_{2j-1}+1) s_{2j-1}^2 \le \sum_{r=1}^{\infty} d_r s_r^2 + \sum_{j=1}^{\infty} s_{2j-1}^2 < \infty \ ,$$

the sequence (u_N) is bounded in $H_+^{1/2}$. Hence, there exists a subsequence converging weakly to some u in $H_+^{1/2}$. In particular, one may assume that (u_N) converges strongly to u in L_+^2.

Since $\|H_{u_N}\| = \rho_1$ is bounded, we infer the strong convergence of operators,

$$\forall h \in L_+^2, H_{u_N}(h) \underset{p\to\infty}{\longrightarrow} H_u(h) \ .$$

We now observe that if ρ^2 is an eigenvalue of $H_{u_N}^2$ of multiplicity m then ρ^2 is an eigenvalue of H_u^2 of multiplicity at most m. Let $(e_N^{(l)})_{1\le l\le m}$ be an orthonormal family of eigenvectors of $H_{u_N}^2$ associated to the eigenvector ρ^2. Let h be in L_+^2 and write

$$h = \sum_{l=1}^{m} (h|e_N^{(l)}) e_N^{(l)} + h_{0,N}$$

where $h_{0,N}$ is the orthogonal projection of h on the orthogonal complement of $E_{u_N}(\rho)$ so that

$$\begin{aligned} \|(H_{u_N}^2 - \rho^2 I)h\|^2 &= \|(H_{u_N}^2 - \rho^2 I)h_{0,N}\|^2 \\ &\ge d_{\rho^2}\|h_{0,N}\|^2 = d_{\rho^2}(\|h\|^2 - \sum_{l=1}^{m} |(h|e_N^{(l)})|^2) \ , \end{aligned}$$

here d_{ρ^2} denotes the distance to the other eigenvalues of $H_{u_N}^2$. By taking the limit as N tends to ∞ one gets

$$\|(H_u^2 - \rho^2 I)h\|^2 \ge d_{\rho^2}(\|h\|^2 - \sum_{l=1}^{m} |(h|e^{(l)})|^2)$$

where $e^{(l)}$ denotes a weak limit of $e_N^{(l)}$. Assume now that the dimension of $E_u(\rho)$ is larger than $m+1$ then we could construct h orthogonal to $(e^{(1)}, \dots e^{(m)})$ with $H_u^2(h) = \rho^2 h$, a contradiction. The same argument allows to obtain that if ρ^2 is not an eigenvalue of $H_{u_N}^2$, ρ^2 is not an eigenvalue of H_u^2.

We now argue as in section 4.1.3.3 above. Let $u_{N,j}$ and $u'_{N,k}$ denote the

orthogonal projections of u_N respectively on $E_{u_N}(\rho_j)$ and on $F_{u_N}(\sigma_k)$ so that we have the orthogonal decompositions,

$$u_N = \sum_{j=1}^N u_{N,j} = \sum_{k=1}^N u'_{N,k}\ .$$

As (u_N) converges strongly in L^2_+, $u_{N,j}$ and $u'_{N,k}$ converge in L^2_+ respectively to some $v^{(j)}$ and to some $v'^{(k)}$ with the identities

$$\rho_j v_j = \Psi_{2j-1} H_u(v_j)\ ,\ H_u^2(v_j) = \rho_j^2 v_j\ ,\ K_u(v'_k) = \sigma_k \Psi_{2k} v'_k\ ,\ K_u^2(v'_k) = \sigma_k^2 v'_k\ .$$

and

$$(u|v_j) = \tau_j^2\ ,\ (u|v'_k) = \kappa_k^2\ .$$

This already implies that v_j, v'_k are not 0, and hence, in view of Lemmas 7 and 8, that

$$\dim E_u(\rho_j) = m_j\ ,\ \dim F_u(\sigma_k) = \ell_k\ .$$

We infer that $u \in \mathcal{V}^{(2)}_{(d_r)_{r\geq 1}}$ and that $\rho_j = s_{2j-1}(u)$, $\sigma_k = s_{2k}(u)$. It remains to identify v_j with the orthogonal projection u_j of u onto $E_u(\rho_j)$, and v'_k with the orthogonal projection u'_k of u onto $F_u(\sigma_k)$. The strategy of passing to the limit, as N tends to infinity, in the decompositions

$$u_N = \sum_{j=1}^N u_{N,j} = \sum_{k=1}^N u'_{N,k}$$

is not easy to apply because of infinite sums. Hence we argue as follows. From the identity

$$\|u_{N,j}\|^2 = (u_N|u_{N,j})$$

we get

$$\|v_j\|^2 = (u|v_j) = (u_j|v_j) = \tau_j^2 = \|u_j\|^2,$$

hence

$$\|v_j - u_j\|^2 = 0\ .$$

Similarly, $v'_k = u'_k$. This completes the proof of the surjectivity, and of the explicit formula (4.2.1). Notice that the convergence is strong in $H^{1/2}$ since the norm of u_N tends to the norm of u.

The continuity of Φ follows as in section 4.1.1. As for the continuity of Φ^{-1}, we argue exactly as for surjectivity above.

$\square$

4.3. Extension to compact Hankel operators

The mapping Φ may be extended to VMO_+ which corresponds to the set of symbols of compact Hankel operators. Namely, let Ω_∞ be the set of sequences $(s_r)_{r\geq 1}$ such that

$$s_1 > s_2 > \cdots > s_n \to 0 \ .$$

Given an arbitrary sequence $(d_r)_{r\geq 1}$ of nonnegative integers, we set

$$\mathcal{V}_{(d_r)_{r\geq 1}} := \Phi^{-1}(\Omega_\infty \times \prod_{r=1}^{\infty} \mathcal{B}_{d_r}) \ .$$

Theorem 7. — *The mapping*

$$\begin{aligned} \Phi : \mathcal{V}_{(d_r)_{r\geq 1}} &\longrightarrow \Omega_\infty \times \prod_{r=1}^{\infty} \mathcal{B}_{d_r} \\ u &\longmapsto ((s_r)_{r\geq 1}, (\Psi_r)_{r\geq 1}) \end{aligned}$$

is a homeomorphism.

Proof. — The proof is the same as before except for the argument of surjectivity in which the boundedness of the sequence (u_N) in $H^{1/2}_+$ does not hold anymore. However, the strong convergence in L^2_+ may be established. The proof is along the same lines as the one developed for Proposition 2 in [13], and is based on the Adamyan-Arov-Krein (AAK)theorem [1], [39]. Let us recall the argument.

First we recall that the AAK theorem states that the $(p+1)$-th singular value of a Hankel operator, as the distance of this operator to operators of rank at most p, is exactly achieved by some Hankel operator of rank at most p, hence, with a rational symbol. We refer to part (2) of the theorem 3.4.1. We set, for every $m \geq 1$,

$$p_m = m + \sum_{r\leq 2m} d_r \ .$$

With the notation of , one easily checks that, for every m,

$$\lambda_{p_{m-1}}(u) > \lambda_{p_m}(u) = \rho_{m+1}(u) \ .$$

By part (1) of the AAK theorem 3.4.1, for every N and every $m = 1, \dots, N$, there exists a rational symbol $u_N^{(m)}$, defining a Hankel operator

of rank p_m, namely $u_N^{(m)} \in \mathcal{V}(2p_m) \cup \mathcal{V}(2p_m - 1)$, such that

$$\|H_{u_N} - H_{u_N^{(m)}}\| = \rho_{m+1}(u_N) = \rho_{m+1}.$$

In particular, we get

$$\|u_N - u_N^{(m)}\|_{L^2} \leq \rho_{m+1}.$$

On the other hand, one has

$$\|H_{u_N^{(m)}}\| \geq \frac{1}{\sqrt{p_m}}(Tr(H^2_{u_N^{(m)}}))^{1/2} \geq \frac{1}{\sqrt{p_m}}\|u_N^{(m)}\|_{H_+^{1/2}}.$$

Hence, for fixed m, the sequence $(u_N^{(m)})_N$ is bounded in $H_+^{1/2}$. Our aim is to prove that the sequence (u_N) is precompact in L_+^2. We show that, for any $\varepsilon > 0$ there exists a finite sequence $v_k \in L_+^2$, $1 \leq k \leq M$ so that

$$\{u_N\}_N \subset \bigcup_{k=1}^{M} B_{L_+^2}(v_k, \varepsilon).$$

Let m be fixed such that $\rho_{m+1} \leq \varepsilon/2$. Since the sequence $(u_N^{(m)})_N$ is uniformly bounded in $H_+^{1/2}$, it is precompact in L_+^2, hence there exists $v_k \in L_+^2$, $1 \leq k \leq M$, such that

$$\{u_N^{(m)}\}_N \subset \bigcup_{k=1}^{M} B_{L_+^2}(v_k, \varepsilon/2)\ .$$

Then, for every N there exists some k such that

$$\|u_N - v_k\|_{L^2} \leq \rho_{m+1} + \|u_N^{(m)} - v_k\|_{L^2} \leq \varepsilon.$$

Therefore $\{u_N\}$ is precompact in L_+^2 and, since u_N converges weakly to u, it converges strongly to u in L_+^2. The proof ends as in the Hilbert-Schmidt case.

The continuity of Φ follows as in section 4.1.1. As for the continuity of Φ^{-1}, we argue exactly as for surjectivity above, except that we have to prove the convergence of u_N to u in VMO_+. This can be achieved exactly as in the proof of Proposition 2 of [13] : the Adamyan-Arov-Krein theorem allows to reduce to the following statement : if $w_N \in \mathcal{V}(2p) \cup \mathcal{V}(2p-1)$ strongly converges to $w \in \mathcal{V}(2p) \cup \mathcal{V}(2p-1)$, then the convergence takes place in VMO — in fact in C^∞. See Lemma 3 of [13]. □

CHAPTER 5

THE SZEGŐ DYNAMICS

This chapter is devoted to the connection between the nonlinear Fourier transform and the Szegő dynamics. In section 5.1, we show that the Szegő evolution has a very simple translation in terms of the nonlinear Fourier transform. Geometric aspects of this evolution law will be discussed in more detail in chapter 7. Then we revisit the classification of traveling waves of the cubic Szegő equation. The last section is devoted to the proof of the almost periodicity of $H^{1/2}$ solutions.

5.1. Evolution under the cubic Szegő flow

5.1.1. The theorem. — In this section, we prove the following result.

Theorem 8. — *Let $u_0 \in H^{1/2}_+$ with*

$$\Phi(u_0) = ((s_r), (\Psi_r)).$$

The solution of

$$i\partial_t u = \Pi(|u|^2 u),\ u(0) = u_0$$

is characterized by

$$\Phi(u(t)) = ((s_r), (\mathrm{e}^{i(-1)^r s_r^2 t}\Psi_r))\ .$$

Remark 2. — *It is in fact possible to define the flow of the cubic Szegő equation on $BMO_+ = BMO(\mathbb{S}^1) \cap L^2_+$, see [18]. The above theorem then extends to the case of an initial datum u_0 in VMO_+ .*

Proof. — In view of the continuity of the flow map on $H^{1/2}_+$, see [11], we may assume that H_{u_0} is of finite rank. Let u be the corresponding solution of the cubic Szegő equation. Let ρ be a singular value of H_u in $\Sigma_H(u)$ such that $m := \dim E_u(\rho) = \dim F_u(\rho) + 1$ and denote by u_ρ the orthogonal projection of u on $E_u(\rho)$. Hence, $u_\rho = 1\!\!1_{\{\rho^2\}}(H_u^2)(u)$. Let us differentiate this equation with respect to time. Recall [11], [14] that

$$\frac{dH_u}{dt} = [B_u, H_u] \text{ with } B_u = \frac{i}{2}H_u^2 - iT_{|u|^2}\ . \tag{5.1.1}$$

Here we recall that T_b denotes the Toeplitz operator of symbol b,

$$T_b(h) = \Pi(bh)\ ,\ h \in L^2_+\ ,\ b \in L^\infty\ .$$

Equation (5.1.1) implies, for every Borel function f,

$$\frac{df(H_u^2)}{dt} = -i[T_{|u|^2}, f(H_u^2)]\ .$$

We get from this Lax pair structure

$$\begin{aligned} \frac{du_\rho}{dt} &= -i[T_{|u|^2}, 1\!\!1_{\{\rho^2\}}(H_u^2)](u) + 1\!\!1_{\{\rho^2\}}(H_u^2)\left(\frac{du}{dt}\right) \\ &= -i[T_{|u|^2}, 1\!\!1_{\{\rho^2\}}(H_u^2)](u) + 1\!\!1_{\{\rho^2\}}(H_u^2)\left(-iT_{|u|^2}u\right)\ , \end{aligned}$$

and eventually

$$\frac{du_\rho}{dt} = -iT_{|u|^2}u_\rho\ . \tag{5.1.2}$$

On the other hand, differentiating the equation

$$\rho u_\rho = \Psi H_u(u_\rho)$$

one obtains

$$\rho\frac{du_\rho}{dt} = \dot\Psi H_u(u_\rho) + \Psi\left([B_u, H_u](u_\rho) + H_u\left(\frac{du_\rho}{dt}\right)\right)$$

Hence, using the expression (5.1.2), we get

$$-i\rho T_{|u|^2}(u_\rho) = \dot\Psi H_u(u_\rho) + \Psi\left(-iT_{|u|^2}H_u(u_\rho) + i\rho^2 H_u(u_\rho)\right)\ ,$$

hence

$$-i[T_{|u|^2}, \Psi]H_u(u_\rho) = (\dot\Psi + i\rho^2\Psi)H_u(u_\rho)\ .$$

We claim that the left hand side of this equality is zero. Assume this claim proved, we get, as $H_u(u_\rho)$ is not identically zero, that $\dot\Psi+i\rho^2\Psi=0$, whence

$$\Psi(t)=\mathrm{e}^{-it\rho^2}\Psi(0)\ .$$

It remains to prove the claim. We first prove that, for any $p\in\mathbb{D}$ such that χ_p is a factor of χ,

$$[T_{|u|^2},\chi_p](e)=0$$

for any $e\in E_u(\rho)$ such that $\chi_p e\in E_u(\rho)$. Recall that

$$\chi_p(z)=\frac{z-p}{1-\overline{p}z}\ .$$

For any L^2 function f,

$$\Pi(\chi_p f)-\chi_p\Pi(f)=K_{\chi_p}(g)=(1-|p|^2)H_{1/(1-\overline{p}z)}(g)\ ,$$

where $\overline{(I-\Pi)f}=Sg$. Consequently, the range of $[\Pi,\chi_p]$ is one dimensional, directed by $\frac{1}{1-\overline{p}z}$. In particular, $[T_{|u|^2},\chi_p](e)$ is proportional to $\frac{1}{1-\overline{p}z}$. On the other hand,

$$\begin{aligned}([T_{|u|^2},\chi_p](e)|1) &= ((T_{|u|^2}(\chi_p e)-\chi_p T_{|u|^2}(e))|1)\\ &= (\chi_p(e)|H_u^2(1))-(\chi_p|1)(e|H_u^2(1))\\ &= (H_u^2(\chi_p(e))|1)-(\chi_p|1)(H_u^2(e)|1)=0\ .\end{aligned}$$

This proves that $[T_{|u|^2},\chi_p](e)=0$.

For the general case, we write $\Psi=\mathrm{e}^{-i\psi}\chi_{p_1}\dots\chi_{p_{m-1}}$ and

$$[T_{|u|^2},\Psi]H_u(u_\rho)=\mathrm{e}^{-i\psi}\sum_{j=1}^{m-1}\prod_{k=1}^{j-1}\chi_{p_k}[T_{|u|^2},\chi_{p_j}]\prod_{k=j+1}^{m-1}\chi_{p_k}H_u(u_\rho)=0\ .$$

It remains to consider the evolution of the Ψ_{2k}'s. Let σ be a singular value of K_u in $\Sigma_K(u)$ such that $\dim F_u(\sigma)=\dim E_u(\sigma)+1$ and denote by u'_σ the orthogonal projection of u onto $F_u(\sigma)$. Recall [14] that

$$\frac{dK_u}{dt}=[C_u,K_u]\ \text{with}\ C_u=\frac{i}{2}K_u^2-iT_{|u|^2}\ .$$

As before, we compute the derivative in time of $u'_\sigma=\mathbb{1}_{\{\sigma^2\}}(K_u^2)(u)$, and get

$$\frac{du'_\sigma}{dt}=-iT_{|u|^2}u'_\sigma\ . \tag{5.1.3}$$

On the other hand, differentiating the equation

$$K_u(u'_\sigma) = \sigma \Psi u'_\sigma$$

one obtains

$$-i[T_{|u|^2}, \Psi]u'_\sigma = (\dot{\Psi} - i\sigma^2\Psi)u'_\sigma \ .$$

As before, we prove that the left hand side of the latter identity is 0, by checking that, for every factor χ_p of Ψ, for any $f \in F_u(\sigma)$ such that $\chi_p f \in F_u(\sigma)$,

$$([T_{|u|^2}, \chi_p](f)|1) = 0 \ .$$

The calculation leads to

$$\begin{aligned}([T_{|u|^2}, \chi_p](f)|1) &= (H_u^2(\chi_p f) - (\chi_p|1)H_u^2(f)|1) \\ &= ((\chi_p - (\chi_p|1))f|u)(u|1),\end{aligned}$$

where we have used (2.1.4). Now $(\chi_p - (\chi_p|1))f \in F_u(\sigma)$ is orthogonal to 1, hence, from Proposition 3, it belongs to $E_u(\sigma)$, hence it is orthogonal to u. This completes the proof.

□

5.2. Application: traveling waves revisited

As an application of Theorems 5 and 6 and of the previous section, we revisit the traveling waves of the cubic Szegő equation. These are the solutions of the form

$$u(t, e^{ix}) = e^{-i\omega t}u_0(e^{i(x-ct)}) \ , \ \omega, c \in \mathbb{R} \ .$$

For $c = 0$, it is easy to see [11] that this condition for $u_0 \in H_+^{1/2}$ corresponds to finite Blaschke product. The problem of characterizing traveling waves with $c \neq 0$ is more delicate, and was solved in [11] by the following result.

Theorem 5.2.1. — [11] *A function u in $H_+^{1/2}$ is a traveling wave with $c \neq 0$ and $\omega \in \mathbb{R}$ if and only if there exist non negative integers ℓ and N, $0 \leq \ell \leq N-1$, $\alpha \in \mathbb{R}$ and a complex number $p \in \mathbb{C}$ with $0 < |p| < 1$ so that*

$$u(z) = \frac{\alpha z^\ell}{1 - pz^N}$$

Here we give an elementary proof of this theorem.

Proof. — The idea is to keep track of the Blaschke products associated to u through the following unitary transform on $L^2(\mathbb{S}^1)$,

$$\tau_\alpha f(\mathrm{e}^{ix}) := f(\mathrm{e}^{i(x-\alpha)}) \ , \ \alpha \in \mathbb{R}.$$

Since τ_α commutes to Π, notice that

$$\tau_\alpha(H_u(h)) = H_{\tau_\alpha(u)}(\tau_\alpha(h)) \ .$$

Consequently, τ_α sends $E_u(\rho)$ onto $E_{\tau_\alpha(u)}(\rho)$, and

$$\tau_\alpha(u_\rho) = [\tau_\alpha(u)]_\rho \ .$$

Applying τ_α to the identity

$$\rho u_\rho = \Psi_\rho H_u(u_\rho) \ ,$$

we infer

$$\rho[\tau_\alpha(u)]_\rho = \tau_\alpha(\Psi_\rho) H_{\tau_\alpha(u)}\left([\tau_\alpha(u)]_\rho\right) \ ,$$

and similarly

$$\rho[\mathrm{e}^{-i\beta}\tau_\alpha(u)]_\rho = \mathrm{e}^{-i\beta}\tau_\alpha(\Psi_\rho) H_{\mathrm{e}^{-i\beta}\tau_\alpha(u)}\left([\mathrm{e}^{-i\beta}\tau_\alpha(u)]_\rho\right) \ .$$

This leads, for every $\rho \in \Sigma_H(u)$, to

$$\Psi_\rho(\mathrm{e}^{-i\beta}\tau_\alpha(u)) = \mathrm{e}^{-i\beta}\tau_\alpha(\Psi_\rho(u)) \ .$$

Applying this identity to $u = u_0$, $\alpha = ct$ and $\beta = \omega t$, and comparing with Theorem 8, we conclude

$$\mathrm{e}^{-it\rho^2}\Psi_\rho(u_0) = \mathrm{e}^{-i\omega t}\tau_{ct}(\Psi_\rho(u_0)) \ .$$

Writing

$$\Psi_\rho(u_0) = \mathrm{e}^{-i\varphi} \prod_{1 \le j \le m-1} \chi_{p_j},$$

we get, for every t,

$$\mathrm{e}^{-it\rho^2} \prod_{1 \le j \le m-1} \chi_{p_j} = \mathrm{e}^{-it(\omega + c(m-1))} \prod_{1 \le j \le m-1} \chi_{\mathrm{e}^{ict}p_j} \ .$$

This imposes, since $c \neq 0$,

$$\rho^2 = \omega + (m-1)c \ , \ p_j = 0 \ ,$$

for every $\rho \in \Sigma_H(u_0)$. In other words, $\Psi_\rho(u_0)(z) = \mathrm{e}^{-i\varphi} z^{m-1}$.

We repeat the same argument for $\sigma \in \Sigma_K(u)$, with $\ell = \dim F_u(\sigma) = \dim E_u(\sigma) + 1$ and

$$K_u(u'_\sigma) = \sigma \Psi_\sigma u'_\sigma \ ,$$

using this time

$$\tau_\alpha(K_u(h)) = \mathrm{e}^{i\alpha} K_{\tau_\alpha(u)}(\tau_\alpha(h)) \ .$$

We get

$$\sigma^2 = \omega - \ell c \ ,$$

and

$$\Psi_\sigma(u_0)(z) = \mathrm{e}^{-i\theta} z^{\ell-1} \ .$$

If we assume that there exists at least two elements $\rho_1 > \rho_2$ in $\Sigma_H(u_0)$, with $m_j = \dim E_{u_0}(\rho_j)$ for $j = 1, 2$, from Lemma 3, there is at least one element σ_1 in $\Sigma_K(u_0)$, satisfying

$$\rho_1 > \sigma_1 > \rho_2.$$

Set $\ell_1 := \dim F_{u_0}(\sigma_1)$, we get

$$(m_1 - 1)c > -\ell_1 c > (m_2 - 1)c$$

which is impossible since m_1, ℓ_1, m_2 are positive integers. Therefore, there is only one element ρ in $\Sigma_H(u_0)$, with $m = \dim E_{u_0}(\rho)$ and at most one element σ in $\Sigma_K(u_0)$, of multiplicity ℓ. Applying the results of section 4.1.2, we obtain

$$u_0(z) = \frac{(\rho^2 - \sigma^2)\mathrm{e}^{-i\varphi}}{\rho} \frac{z^{m-1}}{1 - \frac{\sigma}{\rho}\mathrm{e}^{-i(\varphi+\theta)} z^{\ell+m-1}} \ .$$

This completes the proof. □

5.3. Application to almost periodicity

As a second application of our main result, we prove that the solutions of the Szegő equation are almost periodic. Let us recall some definitions. Let $\mathcal{V}$ be a finite dimensional smooth manifold. A function

$$f : \mathbb{R} \longrightarrow \mathcal{V}$$

is quasi-periodic if there exists a positive integer N, a vector $\omega \in \mathbb{R}^N$, and a continuous function $F : \mathbb{T}^N \to \mathcal{V}$ such that

$$\forall t \in \mathbb{R} \ , \ f(t) = F(\omega t) \ .$$

A similar definition holds for functions valued in a Banach space.
Now let us come to the definition of almost periodic functions. Let X be a Banach space. A function

$$f : \mathbb{R} \longrightarrow X$$

is almost periodic if it is the uniform limit of quasi-periodic functions, namely the uniform limit of finite linear combinations of functions

$$t \longmapsto \mathrm{e}^{i\omega t} x \ ,$$

where $x \in X$ and $\omega \in \mathbb{R}$. Of course, from the explicit formula (1.0.5) and from the evolution under the cubic Szegő flow, for any $u_0 \in \mathcal{V}(d)$, the solution $u(t)$ is quasi-periodic, valued in $\mathcal{V}(d)$, hence valued in every H^s_+. This is also a consequence of the results of [14].
It remains to consider data in $H^{1/2}_+$ corresponding to infinite rank Hankel operators. We are going to use Bochner's criterion, see chapters 1, 2 of [32], namely that $f \in C(\mathbb{R}, X)$ is almost periodic if and only if it is bounded and the set of functions

$$f_h : t \in \mathbb{R} \longmapsto f(t+h) \in X \ , \ h \in \mathbb{R} \ ,$$

is relatively compact in the space of bounded continuous functions valued in X.
Let $u_0 \in \mathcal{V}^{(2)}_{(d_r)_{r\geq 1}}$. Set

$$\Phi(u_0) = ((s_r)_{r\geq 1}, (\Psi_r)_{r\geq 1}) \ .$$

Then, from Theorem 8,

$$\Phi(u(t)) = ((s_r)_{r\geq 1}, (\mathrm{e}^{i(-1)^r s_r^2 t}\Psi_r)_{r\geq 1}) \ .$$

By Theorem 6, it is enough to prove that the set of functions

$$t \in \mathbb{R} \longmapsto \Phi(u(t+h)) \in \mathcal{S}^{(2)}_{(d_r)}$$

is relatively compact in $C(\mathbb{R}, \mathcal{S}^{(2)}_{(d_r)})$. This is equivalent to the relative compactness of the family $(\mathrm{e}^{i(-1)^r s_r^2 h})_{r\geq 1}$ in $(\mathbb{S}^1)^\infty$, $h \in \mathbb{R}$, which is trivial.

CHAPTER 6

LONG TIME INSTABILITY AND UNBOUNDED SOBOLEV ORBITS

The purpose of this chapter is to prove part 2 of Theorem 1, namely that generic data in $C^\infty_+(\mathbb{S}^1)$ generate superpolynomially unbounded trajectories of the Szegő evolution in every Sobolev space H^s, $s > \frac{1}{2}$. This of course is in strong contrast with the compactness properties established for the same trajectories in $H^{\frac{1}{2}}$ in the previous chapter. The proof takes advantage of the nonlinear Fourier transform constructed in the previous chapters, by proving superpolynomial long time instability for quasiperiodic solutions. Again, the key idea is collapsing singular values.

6.1. Instability and genericity of unbounded orbits

In this section, we show that part 2 of Theorem 1 is a consequence of the following long time instability result. We denote by d_∞ a distance function on $C^\infty_+(\mathbb{S}^1)$ which defines the C^∞ topology.

Theorem 9. — *For any $v \in C^\infty_+$, for any M, for any $s > \frac{1}{2}$, there exists a sequence $(v^{(n)})$ of elements of C^∞_+ tending to v in C^∞_+ and sequences of times $(\overline{t}_n)$, $(\underline{t}^n)$, tending to ∞, such that*

$$\frac{\|Z(\overline{t}_n)v^{(n)}\|_{H^s}}{|\overline{t}_n|^M} \underset{n\to\infty}{\longrightarrow} \infty \ ,$$

and

$$d_\infty(Z(\underline{t}^n)v^{(n)}, v^{(n)}) \underset{n\to\infty}{\longrightarrow} 0 \ .$$

Assuming Theorem 9, a Baire category argument leads to the following proof of part 2 of Theorem 1.

Proof. — For every positive integer M, we denote by $\mathcal{O}_M$ the set of functions $v \in C^\infty_+$ such that there exist $\bar{t}, \underline{t}$ with $|\bar{t}| > M, |\underline{t}| > M$, and

$$\|Z(\bar{t})v\|_{H^{\frac12+\frac1M}} > M|\bar{t}|^M, \ , \ d_\infty(Z(\underline{t})v, v) < \frac{1}{M} \ .$$

From the global wellposedness theory [11], $\mathcal{O}_M$ is an open subset of C^∞_+. Furthermore, Theorem 9 implies that $\mathcal{O}_M$ is dense. From the Baire theorem applied to the Fréchet space C^∞_+, we conclude that the countable intersection

$$\mathscr{G} := \cap_{M\geq 1}\mathcal{O}_M$$

is a dense G_δ subset. $\square$

6.2. A family of quasiperiodic solutions

Let us come to the proof of Theorem 9. Our strategy is the following. First of all, rational functions are dense in C^∞_+. Furthermore, from Theorem 7.1 in [11], in the finite dimensional manifold of rational functions with associated Hankel operators of given rank q, those functions u for which the singular values of H_u and K_u are simple, is an open dense subset. From Theorem 2, every such rational function reads $v := u(\mathbf{s}, \mathbf{\Phi})$ for some finite sequence $\mathbf{s}$ of positive numbers of length $2q$ or $2q-1$, and some sequence $\mathbf{\Phi}$, of the same length, of complex numbers of modulus 1. Up to adding a small positive number to the sequence $\mathbf{s}$, we infer that those functions $v := u(\mathbf{s}, \mathbf{\Phi})$, with $\mathbf{s} \in \Omega_{2q}$ and $\mathbf{\Phi} \in (\mathbb{S}^1)^{2q}$, are dense in C^∞_+. Therefore it is enough to prove the statement of Theorem 9 if v is a rational function $v := u(\mathbf{s}, \mathbf{\Phi})$ with

$$\mathbf{s} = (\rho_1, \sigma_1, \dots \rho_q, \sigma_q), \ \mathbf{\Phi} = (\Phi_j)_{1\leq j\leq 2q} \in (\mathbb{S}^1)^{2q}$$

where

$$\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \cdots > \rho_q > \sigma_q > 0.$$

We are going to construct a sequence $(v^{(n)})$ of elements of C^∞_+ tending to v in C^∞_+ and a sequence of times $(\bar{t}_n)$, such that

$$\frac{\|Z(\bar{t}_n)v^{(n)}\|_{H^s}}{|\bar{t}_n|^M} \underset{n\to\infty}{\longrightarrow} \infty \ ,$$

with

$$v^{(n)} = u(\mathbf{s}^{(n)}, \mathbf{\Phi}^{(n)}) \ ,$$

where $\mathbf{s}^{(n)}, \mathbf{\Phi}^{(n)}$ are to be chosen. At this stage, we remark that the existence of the second sequence $(\underline{t}^n)$ comes for free. Indeed, for a given n, we already noticed that the mapping $t \mapsto Z(t)v^{(n)}$ is quasi-periodic valued into some manifold $\mathcal{V}(d_n)$, continuously imbedded into C^∞_+. Hence the function

$$t \in \mathbb{R} \mapsto d_\infty(Z(t)v^{(n)}, v^{(n)})$$

is quasiperiodic valued in $\mathbb{R}$. We then use a classical property, namely, if f is a quasiperiodic function, for every $\varepsilon > 0$, there exists infinitely many $t \in \mathbb{R}$ such that

$$|f(t) - f(0)| < \varepsilon \ .$$

The existence of $\underline{t}^n$ follows.

Let us come back to the construction of the sequences $(v^{(n)})$ and $(\bar{t}_n)$. For technical reasons, it is more convenient to start from the construction of a singular sequence $u^{(n)}$, which will play the role of $Z(\bar{t}_n)v^{(n)}$ in Theorem 9, and then to check that $Z(-\bar{t}_n)u^{(n)}$ has the desired limit v in C^∞_+. We introduce the following class of rational functions.

Let $N \geq 2$ be an integer. Denote by $\mathcal{X}_N$ the subset of $(\xi, \eta) \in \mathbb{R}^{2N-1}$ such that $\xi = (\xi_1, \dots, \xi_N)$, $\eta = (\eta_1, \dots, \eta_{N-1})$ and

$$\xi_1 > \eta_1 > \xi_2 > \eta_2 > \dots \eta_{N-1} > \xi_N > 0. \tag{6.2.1}$$

Given $\mathbf{\Psi} = (\Psi_r)_{1\leq r\leq 2q} \in (\mathbb{S}^1)^{2q}$, we consider the family $u^{\delta,\varepsilon}$ for $\delta, \varepsilon \to 0$ with

$$u^{\delta,\varepsilon} = u\left((\mathbf{s}, \delta(1+\varepsilon\xi_1), \delta(1+\varepsilon\eta_1), \dots, \delta(1+\varepsilon\xi_N), 0\right), (\mathbf{\Psi}, 1, \dots, 1)) \,.$$

From the explicit formula 1.0.5, we have

$$u^{\delta,\varepsilon}(z) = \left\langle \mathscr{C}_{\delta,\varepsilon}{}^{-1}(z) \begin{pmatrix} \mathbf{\Psi}_q \\ 1_N \end{pmatrix}, \begin{pmatrix} 1_q \\ 1_N \end{pmatrix} \right\rangle \tag{6.2.2}$$

where $\mathbf{\Psi}_q = (\Psi_{2a-1})_{1\le a\le q}$,

$$1_N = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^N$$

and

$$\mathscr{C}_{\delta,\varepsilon}(z) = \begin{pmatrix} \mathscr{E}(z) & \mathscr{A}_{\delta,\varepsilon}(z) \\ \\ \mathscr{B}_{\delta,\varepsilon}(z) & \frac{1}{\delta}\mathscr{C}_\varepsilon(z) \end{pmatrix}$$

with

$$\mathscr{E}(z) = \left(\frac{\rho_a - \sigma_b z \Psi_{2a-1}\Psi_{2b}}{\rho_a^2 - \sigma_b^2}\right)_{1\le a,b\le q},$$

$$\mathscr{A}_{\delta,\varepsilon}(z) = \left(\left(\frac{\rho_a - \delta z(1+\varepsilon\eta_k)\Psi_{2a-1}}{\rho_a^2 - \delta^2(1+\varepsilon\eta_k)^2}\right)_{\substack{1\le a\le q\\1\le k\le N-1}}, \frac{1}{\rho_a}{}_{1\le a\le q}\right)$$

$$\mathscr{B}_{\delta,\varepsilon}(z) = \left(\frac{\delta(1+\varepsilon\xi_j) - \sigma_b z\Psi_{2b}}{\delta^2(1+\varepsilon\xi_j)^2 - \sigma_b^2}\right)_{\substack{1\le j\le N\\1\le b\le q}}$$

$\mathscr{C}_\varepsilon(z) = (c_{\varepsilon,jk}(z))_{1\le j,k\le N}$, with

$$\begin{aligned} c_{\varepsilon,jk}(z) &:= \frac{1+\varepsilon\xi_j - z(1+\varepsilon\eta_k)}{(1+\varepsilon\xi_j)^2 - (1+\varepsilon\eta_k)^2}\ ,\ 1\le k\le N-1\ ; \\ c_{\varepsilon,jN}(z) &:= \frac{1}{1+\varepsilon\xi_j}\ ,\ 1\le j\le N\ . \end{aligned}$$

The following proposition is the key of the proof of Theorem 9.

Proposition 4. — *There exists a nonempty open subset $\mathcal{X}'_N$ of $\mathcal{X}_N$ such that, for every $(\xi,\eta)\in\mathcal{X}'_N$, the following properties hold.*

1. *The solution of the cubic Szegő equation with initial datum $u^{\delta,\varepsilon}$ at time $\frac{1}{2\varepsilon\delta^2}$ satisfies*

$$Z\left(\frac{1}{2\varepsilon\delta^2}\right)u^{\delta,\varepsilon} = u(\mathbf{s}, \tilde{\mathbf{\Psi}}_{\delta,\varepsilon}) + o(1)$$

in C^∞_+ as ε and δ tend to 0, where

$$\tilde{\Psi}_{\delta,\varepsilon,2a-1} = \Psi_{2a-1}\mathrm{e}^{-i\frac{\rho_a^2}{2\varepsilon\delta^2}}\ ,\ \tilde{\Psi}_{\delta,\varepsilon,2b} = \Psi_{2b}\mathrm{e}^{i\frac{\sigma_b^2}{2\varepsilon\delta^2}}\ .$$

2. *As ε and δ tend to 0 with $\varepsilon \ll \delta$,*

$$\forall s \in (\frac{1}{2}, 1), \exists C_s > 0 : \|u^{\delta,\varepsilon}\|_{H^s} \geq C_s \frac{\delta}{\varepsilon^{(N-1)(2s-1)}}.$$

Let us show how Theorem 9 is a consequence of Proposition 4. Let M be a positive integer and $\mathbf{\Phi} \in (\mathbb{S}^1)^{2q}$, and $s > \frac{1}{2}$. We may assume that $s < 1$. Choose an integer N such that $(N-1)(2s-1) \geq 2M+1$, and $(\xi, \eta) \in \mathcal{X}'_N$. Consider

$$u^{(n)} = u^{\delta_n, \delta_n^2} ,$$

where δ_n is a sequence tending to 0 such that

$$\mathrm{e}^{-i\frac{\rho_a^2}{2\delta_n^4}} \to \mathrm{e}^{-i\varphi_a} \ , \ \mathrm{e}^{i\frac{\sigma_b^2}{2\delta_n^4}} \to \mathrm{e}^{i\theta_b} \ ,$$

and where $\mathbf{\Psi}$ is defined as

$$\Psi_{2a-1} = \Phi_{2a-1}\mathrm{e}^{i\varphi_a} \ , \ \Psi_{2b} = \Phi_{2b}\mathrm{e}^{-i\theta_b} \ .$$

Then Proposition 4 implies that

1. $v^{(n)} := Z\left(\frac{1}{2\delta_n^4}\right) u^{(n)}$ tends to $v = u(\mathbf{s}, \mathbf{\Phi})$ in C^∞_+.
2. The following estimates hold,

$$\left\| Z\left(-\frac{1}{2\delta_n^4}\right) v^{(n)} \right\|_{H^s} = \|u^{(n)}\|_{H^s} \geq C_s \frac{1}{\delta_n^{2(N-1)(2s-1)-1}} \gg \left(\frac{1}{2\delta_n^4}\right)^M .$$

This proves Theorem 9 with $\overline{t}_n = -\frac{1}{2\delta_n^4}$.

The proof of Proposition 4 requires several steps, which will be achieved in the two next sections.

6.3. Construction of the smooth family of data

We first address the first part of Proposition 4. From Theorem 2,

$$Z\left(\frac{1}{2\varepsilon\delta^2}\right) u^{\delta,\varepsilon}(z) = \left\langle \tilde{\mathscr{C}}_{\delta,\varepsilon}^{-1}(z) \begin{pmatrix} \mathbf{\Psi}_q \\ \mathbf{\Psi}_{\varepsilon N} \end{pmatrix}, \begin{pmatrix} 1_q \\ 1_N \end{pmatrix} \right\rangle$$

where

$$\begin{aligned} \mathbf{\Psi}_q &:= (\Psi_{2a-1}\mathrm{e}^{-i\frac{\rho_a^2}{2\varepsilon\delta^2}})_{1\leq a\leq q} \\ \mathbf{\Psi}_{\varepsilon N} &:= (\mathrm{e}^{-i\frac{(1+\varepsilon\xi_j)^2}{2\varepsilon}})_{1\leq j\leq N} = \mathrm{e}^{-i\frac{1}{2\epsilon}}\left(\mathrm{e}^{-i\xi_j(1+\varepsilon\frac{\xi_j}{2})}\right)_{1\leq j\leq N} \end{aligned}$$

and

$$\tilde{\mathscr{C}}_{\delta,\varepsilon}(z) = \begin{pmatrix} \tilde{\mathscr{E}}_{\delta,\varepsilon}(z) & \tilde{\mathscr{A}}_{\delta,\varepsilon}(z) \\ \\ \tilde{\mathscr{B}}_{\delta,\varepsilon}(z) & \frac{1}{\delta}\tilde{\mathscr{C}}_{\varepsilon}(z) \end{pmatrix}$$

with

$$\tilde{\mathscr{E}}_{\delta,\varepsilon}(z) = \left(\frac{\rho_a - \sigma_b z \Psi_{2a-1}\Psi_{2b} \mathrm{e}^{-i(\frac{\rho_a^2-\sigma_b^2}{2\varepsilon\delta^2})}}{\rho_a^2 - \sigma_b^2} \right)_{1\le a,b\le q},$$

$$\tilde{\mathscr{A}}_{\delta,\varepsilon}(z) = \left(\left(\frac{\rho_a - \delta z(1+\varepsilon\eta_k)\Psi_{2a-1}\mathrm{e}^{-i(\frac{\rho_a^2-\delta^2(1+\varepsilon\eta_k)^2}{2\varepsilon\delta^2})}}{\rho_a^2 - \delta^2(1+\varepsilon\eta_k)^2} \right)_{\substack{1\le a\le q\\ 1\le k\le N-1}}, \left(\frac{1}{\rho_a}\right)_{1\le a\le q} \right)$$

$$\tilde{\mathscr{B}}_{\delta,\varepsilon}(z) = \left(\frac{\delta(1+\varepsilon\xi_j) - \sigma_b z \Psi_{2b}\mathrm{e}^{-i(\frac{\delta^2(1+\varepsilon\xi_j)^2-\sigma_b^2}{2\varepsilon\delta^2})}}{\delta^2(1+\varepsilon\xi_j)^2 - \sigma_b^2} \right)_{\substack{1\le j\le N\\ 1\le b\le q}}$$

and

$$\tilde{\mathscr{C}}_{\varepsilon}(z) = \left(\frac{1+\varepsilon\xi_j - z(1+\varepsilon\eta_k)\mathrm{e}^{-i(\xi_j-\eta_k)-i\varepsilon\frac{\xi_j^2-\eta_k^2}{2}}}{\varepsilon(\xi_j-\eta_k)(2+\varepsilon(\xi_j+\eta_k))}, \frac{1}{1+\varepsilon\xi_j} \right).$$

Remark that from Theorem 5, the functions

$$u(\mathbf{s}, (\Psi_r \mathrm{e}^{-i\psi_r})_{1\le r\le 2q}) \ , \ (\psi_r) \in \mathbb{T}^{2q} \ ,$$

lie in a compact subset — a torus — of the manifold $\mathcal{V}(2q)$. This implies that the distance of the zeroes of the denominator $\det \tilde{\mathscr{E}}_{\delta,\varepsilon}(z)$ to the closed unit disc is bounded from below by a positive constant. As a consequence, the matrices $\tilde{\mathscr{E}}_{\delta,\varepsilon}(z)$ are invertible with a bounded inverse, for every z in a fixed neighborhood of $\overline{\mathbb{D}}$. The first part of Proposition 4 is a consequence of the following lemma.

Lemma 13. — *There exists a nonempty open subset $\mathcal{X}_N''$ of $\mathcal{X}_N$, such that, for every $(\xi,\eta) \in \mathcal{X}_N''$, there exist $r_{\xi,\eta} > 1$ and $\varepsilon_{\xi,\eta} > 0$ such that, for $0 < \varepsilon < \varepsilon_{\xi,\eta}$, the matrix $\tilde{\mathscr{C}}_{\varepsilon}(z)$ is invertible with a bounded inverse for every z such that $|z| \le r_{\xi,\eta}$.*

Let us admit this lemma for a while. Using it, we can easily describe the inverse of the matrix $\tilde{\mathscr{C}}_{\delta,\varepsilon}(z)$.

Lemma 14. — *For every* $(\xi,\eta)\in\mathcal{X}''_N$*, there exist* $r_{\xi,\eta}>1$ *and* $\gamma_{\xi,\eta}>0$ *, such that, for every* z *with* $|z|\le r_{\xi,\eta}$*, for every* $\delta,\varepsilon<\gamma_{\xi,\eta}$*, the matrix*

$$\tilde{\mathscr{J}}_{\delta,\varepsilon}(z):=\tilde{\mathscr{E}}_{\delta,\varepsilon}(z)-\delta\tilde{\mathscr{A}}_{\delta,\varepsilon}(z)\tilde{\mathscr{C}}_{\varepsilon}^{-1}(z)\tilde{\mathscr{B}}_{\delta,t_\varepsilon}(z),$$

is invertible with a bounded inverse . The inverse of the matrix $\tilde{\mathscr{C}}_{\delta,\varepsilon}(z)$ *is given by*

$$(\tilde{\mathscr{C}}_{\delta,\varepsilon})^{-1}=\begin{pmatrix}\tilde{\mathscr{J}}_{\delta,\varepsilon}^{-1} & -\delta\,\tilde{\mathscr{J}}_{\delta,\varepsilon}^{-1}\tilde{\mathscr{A}}_{\delta,\varepsilon}\tilde{\mathscr{C}}_{\varepsilon}^{-1}\\ -\delta\tilde{\mathscr{C}}_{\varepsilon}^{-1}\tilde{\mathscr{B}}_{\delta,\varepsilon}\,\tilde{\mathscr{J}}_{\delta,\varepsilon}^{-1} & \delta\tilde{\mathscr{C}}_{\varepsilon}^{-1}[I+\delta\tilde{\mathscr{B}}_{\delta,\varepsilon}\,\tilde{\mathscr{J}}_{\delta,\varepsilon}^{-1}\tilde{\mathscr{A}}_{\delta,\varepsilon}\tilde{\mathscr{C}}_{\varepsilon}^{-1}]\end{pmatrix}.$$

Furthermore,

$$Z\left(\frac{1}{2\varepsilon\delta^2}\right)u^{\delta,\varepsilon}=u(\mathbf{s},\tilde{\mathbf{\Psi}}_{\delta,\varepsilon})+o(1)$$

in C^∞_+ *as* ε *and* δ *tend to* 0*, where*

$$\tilde{\Psi}_{\delta,\varepsilon,2j}=\Psi_{2j}\mathrm{e}^{-i\frac{\rho_j^2}{2\varepsilon\delta^2}}\ ,\ \tilde{\Psi}_{\delta,\varepsilon,2k-1}=\Psi_{2k-1}\mathrm{e}^{i\frac{\sigma_k^2}{2\varepsilon\delta^2}}\ .$$

Proof. — The invertibility of the matrix $\tilde{\mathscr{J}}_{\delta,\varepsilon}(z)$ for δ and ε small enough and $|z|<r_{\xi,\eta}$ comes from the already observed invertibility of $\tilde{\mathscr{E}}_{\delta,\varepsilon}(z)$ in a neighborhood of $\overline{D}$, with a bounded inverse, and from Lemma 13. In the next calculations, we drop the variable z for simplicity.
Write, for $X_q,X'_q\in\mathbb{R}^q$, $Y_N,Y'_N\in\mathbb{R}^N$,

$$\tilde{\mathscr{C}}_{\delta,\varepsilon}\begin{pmatrix}X_q\\ Y_N\end{pmatrix}=\begin{pmatrix}X'_q\\ Y'_N\end{pmatrix}$$

so that

$$\begin{aligned}\tilde{\mathscr{E}}_{\delta,\varepsilon}X_q+\tilde{\mathscr{A}}_{\delta,\varepsilon}Y_N &= Y'_q\\ \tilde{\mathscr{B}}_{\delta,\varepsilon}X_q+\frac{1}{\delta}\tilde{\mathscr{C}}_{\varepsilon}Y_N &= Y'_N.\end{aligned}$$

and solve this system to get

$$\begin{aligned}Y_N &= \delta\tilde{\mathscr{C}}_{\varepsilon}^{-1}(Y'_N-\tilde{\mathscr{B}}_{\delta,\varepsilon}X_q)\\ \tilde{\mathscr{J}}_{\delta,\varepsilon}X_q &= Y'_q-\delta\tilde{\mathscr{A}}_{\delta,\varepsilon}\tilde{\mathscr{C}}_{\varepsilon}^{-1}(Y'_N)\end{aligned}$$

and eventually

$$\begin{aligned}X_q &= \tilde{\mathscr{J}}_{\delta,\varepsilon}^{-1}(Y'_q)-\delta\,\tilde{\mathscr{J}}_{\delta,\varepsilon}^{-1}\tilde{\mathscr{A}}_{\delta,\varepsilon}\tilde{\mathscr{C}}_{\varepsilon}^{-1}(Y'_N)\\ Y_N &= \delta\tilde{\mathscr{C}}_{\varepsilon}^{-1}[Y'_N+\delta\tilde{\mathscr{B}}_{\delta,\varepsilon}\tilde{\mathscr{J}}_{\delta,\varepsilon}^{-1}\tilde{\mathscr{A}}_{\delta,\varepsilon}\tilde{\mathscr{C}}_{\varepsilon}^{-1}(Y'_N)-\tilde{\mathscr{B}}_{\delta,\varepsilon}\tilde{\mathscr{J}}_{\delta,\varepsilon}^{-1}(Y'_q)].\end{aligned}$$

It gives the formula for the inverse. Furthermore, as δ, ε tend to 0, we obtain

$$(\tilde{\mathscr{C}}_{\delta,\varepsilon})^{-1} = \begin{pmatrix} \tilde{\mathscr{E}}_{\delta,\varepsilon}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + o(1).$$

This completes the proof of Lemma 14. □

The proof of the first part of Proposition 4 is thus reduced to the proof of Lemma 13, which we now begin.

Proof. — Recall that

$$\tilde{\mathscr{C}}_\varepsilon(z) = \left(\frac{1+\varepsilon\xi_j - z(1+\varepsilon\eta_k)\mathrm{e}^{-i(\xi_j-\eta_k)-i\varepsilon\frac{\xi_j^2-\eta_k^2}{2}}}{\varepsilon(\xi_j-\eta_k)(2+\varepsilon(\xi_j+\eta_k))}, \frac{1}{1+\varepsilon\xi_j} \right)_{1\le j\le N, 1\le k\le N-1}.$$

From Cramer's formulae,

$$\tilde{\mathscr{C}}_\varepsilon(z)^{-1} = \frac{1}{\det \tilde{\mathscr{C}}_\varepsilon(z)} {}^t\mathrm{Co}\tilde{\mathscr{C}}_\varepsilon(z),$$

where Co denotes the matrix of cofactors. As the coefficients of $\tilde{\mathscr{C}}_\varepsilon(z)$ are polynomials in z of degree at most 1, the coefficients of $\mathrm{Co}\tilde{\mathscr{C}}_\varepsilon(z)$ are polynomial in z of degree at most $N-1$. Moreover, these coefficients grow at most as $\frac{1}{\varepsilon^{N-1}}$. Hence, it suffices to prove that, for (ξ, η) in a suitable non empty open subset of $\mathcal{X}_N$, the family of polynomials $\varepsilon^{N-1} \det \tilde{\mathscr{C}}_\varepsilon(z)$ converges, as ε tends to zero, to a polynomial of degree $N-1$ having all its roots outside a disc of radius $r_{\xi,\eta} > 1$.
Let us compute

$$\begin{aligned} P(\xi,\eta)(z) &= \lim_{\varepsilon\to 0}(2\varepsilon)^{N-1} \det \tilde{\mathscr{C}}_\varepsilon(z) \\ &= \det \left(\left(\frac{1 - z\mathrm{e}^{-i(\xi_j-\eta_k)}}{\xi_j-\eta_k} \right)_{1\le j\le N; 1\le k\le N-1}, 1_N \right). \end{aligned}$$

Notice that the determinant in the right hand side is a polynomial in z of degree $N-1$ whose coefficient of z^{N-1} equals

$$(-1)^{N-1} \left(\prod_k \mathrm{e}^{i\eta_k} \right) \det \left(\left(\frac{\mathrm{e}^{-i\xi_j}}{\xi_j-\eta_k} \right)_{1\le j\le N; 1\le k\le N-1}, 1_N \right).$$

With the choice $\xi = \xi^* = (2\pi(N-j+1))_{1\leq j\leq N}$, this determinant is

$$(-1)^{N-1}\left(\prod_k \mathrm{e}^{i\eta_k}\right)\det\left(\left(\frac{1}{\xi_j-\eta_k}\right)_{1\leq j\leq N;1\leq k\leq N-1}, 1_N\right).$$

Developing this determinant with respect to the last column, we are led to compute Cauchy determinants. Let us recall that a Cauchy matrix is a matrix of the form $\left(\dfrac{1}{a_j+b_k}\right)$. Its determinant is given by

$$\det\left(\frac{1}{a_j+b_k}\right) = \frac{\prod_{i<j}(a_i-a_j)\prod_{k<l}(b_k-b_l)}{\prod_{j,k}(a_j+b_k)}. \tag{6.3.1}$$

In view of this formula, we get

$$\begin{aligned}
&\det\left(\left(\frac{1}{\xi_j-\eta_k}\right)_{1\leq j\leq N;1\leq k\leq N-1}, 1_N\right) = \\
&\sum_{r=1}^{N}(-1)^{r+N}\frac{\prod_{i<j,i,j\neq r}(\xi_i-\xi_j)\prod_{k<\ell}(\eta_k-\eta_\ell)}{\prod_{j\neq r,k}(\xi_j-\eta_k)}\\
=\ &\frac{\sum_{r=1}^{N}(-1)^{r+N}\prod_{1\leq k\leq N-1}(\xi_r-\eta_k)\prod_{i<j,i,j\neq r}(\xi_i-\xi_j)\prod_{k<\ell}(\eta_k-\eta_\ell)}{\prod_{j,k}(\xi_j-\eta_k)}
\end{aligned}$$

and we observe that every term in the sum is different from 0 and has the sign of $(-1)^{N-1}$. Therefore this quantity is not zero. On the other hand, Theorem 2 tells us that the roots of $\det\widetilde{\mathscr{C}_\varepsilon}(z)$ are located outside the unit disc. Hence all the roots of $P(\xi,\eta)$ belong to $\{z,\ |z|\geq 1\}$. Furthermore,

$$P(\xi^*,\eta)(z) = \det\left(\left(\frac{1}{\xi_j-\eta_k}\right)_{1\leq j\leq N;1\leq k\leq N-1}, 1_N\right)\prod_{k=1}^{N-1}(1-z\mathrm{e}^{i\eta_k}).$$

The above calculation shows that $P(\xi^*,\eta)$ has $N-1$ zeroes which belong to the unit circle, namely $\mathrm{e}^{-i\eta_k}, k=1,\dots,N-1$. Fix $\eta^*\in\mathbb{R}^{N-1}$ such that $(\xi^*,\eta^*)\in\mathcal{X}_N$ and $\eta^*_k\neq\eta^*_l \bmod 2\pi$ for every $k\neq l$, so that these $N-1$ zeroes are simple if $\eta=\eta^*$. By the implicit function theorem, there exist $\alpha>0$ such that, if $|\xi-\xi^*|<\alpha$ and $|\eta-\eta^*|<\alpha$, the polynomial $P(\xi,\eta)$ has $N-1$ simple zeroes $\{z_k(\xi,\eta);\ 1\leq k\leq N-1\}$. The first part of Proposition 4 is a direct consequence of the following lemma.

Lemma 15. — *There exists a nonempty open subset $\mathcal{X}_N''$ of $\mathcal{X}_N$ such that, for every $(\xi,\eta)\in\mathcal{X}_N''$, for $k=1,\dots,N-1$, $|z_k(\xi,\eta)|>1$.*

Proof. — The functions $\xi\mapsto z_k(\xi,\eta)$ are analytic and satisfy

$$|z_k(\xi,\eta)|^2\geq 1\ ,\ |z_k(\xi^*,\eta)|^2=1\ .$$

Denote by $Q_k(\xi,\eta)$ the Hessian quadratic form of the function

$$\xi\mapsto|z_k(\xi,\eta)|^2\ .$$

We know that $Q_k(\xi^*,\eta)$ is a nonnegative quadratic form. We claim that, for any k, $Q_k(\xi^*,\eta)$ is not 0 for η in a dense open subset of the ball of radius α centered at η^*. If this claim is correct, then, for such η, for ξ close enough to ξ^* such that $\xi-\xi^*$ does not belong to the union of the kernels of $Q_1(\xi^*,\eta),\dots,Q_{N-1}(\xi^*,\eta)$, we have $|z_k(\xi,\eta)|>1$ for $k=1,\dots,N-1$. This provides us with the nonempty open subset $\mathcal{X}_N''$ in the lemma.
Therefore it suffices to prove that the Laplacian of $\xi\mapsto|z_k(\xi,\eta)|^2$, which coincides with the trace of $Q_k(\xi,\eta)$, is not 0 identically in η for $\xi=\xi^*$. Let us compute $\Delta(|z_k|^2)(\xi^*,\eta)$.

$$\sum_{j=1}^N\frac{\partial^2}{\partial\xi_j^2}\left(\frac{|z_k|^2}{2}\right)_{|\xi=\xi^*}=\sum_{j=1}^N\left(\mathrm{Re}\left(\overline{z}_k\frac{\partial^2 z_k}{\partial\xi_j^2}\right)_{|\xi=\xi^*}+\left|\frac{\partial z_k}{\partial\xi_j}\right|^2_{|\xi=\xi^*}\right)\ .$$

Differentiating the equation $P(\xi,\eta)(z_k(\xi,\eta))=0$, we obtain

$$\frac{\partial z_k}{\partial\xi_j}=-\frac{\frac{\partial P}{\partial\xi_j}}{\frac{\partial P}{\partial z}}\ ,\ \frac{\partial^2 z_k}{\partial\xi_j^2}=\frac{-\frac{\partial^2 P}{\partial\xi_j^2}}{\frac{\partial P}{\partial z}}+2\frac{\frac{\partial^2 P}{\partial\xi_j\partial z}\frac{\partial P}{\partial\xi_j}}{\left(\frac{\partial P}{\partial z}\right)^2}-\frac{\left(\frac{\partial P}{\partial\xi_j}\right)^2\frac{\partial^2 P}{\partial z^2}}{\left(\frac{\partial P}{\partial z}\right)^3}\ .$$

Introduce the following quantities.

$$\begin{aligned}
D_{jk}\ &:=\ \frac{1}{\xi_j^*-\eta_k}\det\left(\left(\frac{1}{\xi_r^*-\eta_l}\right)_{r\neq j,l\neq k},1_{N-1}\right),\\
D\ &:=\ \det\left(\left(\frac{1}{\xi_r^*-\eta_l}\right)_{1\leq r\leq N,1\leq l\leq N-1},1_N\right)=\sum_{j=1}^N(-1)^{j+k}D_{jk}\ ,\ k=1,\dots,N-1,\\
\zeta_k\ &:=\ \prod_{l\neq k}(1-\mathrm{e}^{i(\eta_l-\eta_k)})\ ,\\
a_{lk}\ &:=\ -\frac{\mathrm{e}^{i(\eta_l-\eta_k)}}{1-\mathrm{e}^{i(\eta_l-\eta_k)}}=\frac{\mathrm{e}^{i\frac{(\eta_l-\eta_k)}{2}}}{2i\sin\frac{(\eta_l-\eta_k)}{2}}.
\end{aligned}$$

Notice that $\mathrm{Re}(a_{lk}) = \frac{1}{2}$.
Differentiating

$$P(\xi,\eta)(z) = \det\left(\left(\frac{1 - z\mathrm{e}^{-i(\xi_j-\eta_k)}}{\xi_j - \eta_k}\right)_{1\le j\le N; 1\le k\le N-1}, 1_N\right),$$

one gets, after some computations,

$$\begin{aligned}
\frac{\partial P}{\partial \xi_j}_{|\xi=\xi^*, z=z_k} &= i(-1)^{j+k} D_{jk}\zeta_k \ , \ \frac{\partial P}{\partial z}_{|\xi=\xi^*, z=z_k} = -\mathrm{e}^{i\eta_k}\zeta_k D \ , \\
\frac{\partial^2 P}{\partial \xi_j^2}_{|\xi=\xi^*, z=z_k} &= (-1)^{j+k}\zeta_k D_{jk}\left(1 - \frac{2i}{\xi_j^* - \eta_k}\right) \ , \\
\frac{\partial^2 P}{\partial z^2}_{|\xi=\xi^*, z=z_k} &= 2\sum_{l\neq k} \frac{\zeta_k \mathrm{e}^{i(\eta_l+\eta_k)}}{1 - \mathrm{e}^{i(\eta_l-\eta_k)}} D = -2\zeta_k \mathrm{e}^{2i\eta_k}\sum_{l\neq k} a_{lk} D \ ,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 P}{\partial \xi_j \partial z}_{|\xi=\xi^*, z=z_k} &= (-1)^{j+k}\zeta_k D_{jk}\mathrm{e}^{i\eta_k}\left(i + \frac{1}{\xi_j^* - \eta_k} - i\sum_{l\neq k}\frac{\mathrm{e}^{i(\eta_l-\eta_k)}}{1 - \mathrm{e}^{i(\eta_l-\eta_k)}}\right) + \\
+ \ & \zeta_k \mathrm{e}^{i\eta_k}\sum_{l\neq k}(-1)^{j+l} D_{jl}\left(\frac{1}{\xi_j^* - \eta_l} - i\frac{\mathrm{e}^{i(\eta_l-\eta_k)}}{1 - \mathrm{e}^{i(\eta_l-\eta_k)}}\right) \\
= \ & \zeta_k \mathrm{e}^{i\eta_k}\left((-1)^{j+k} D_{jk}\left(i + \frac{1}{\xi_j^* - \eta_k} + i\sum_{l\neq k} a_{lk}\right) + \sum_{l\neq k}(-1)^{j+l} D_{jl}\left(\frac{1}{\xi_j^* - \eta_l} + i a_{lk}\right)\right) \ .
\end{aligned}$$

Hence, inserting these formulae to compute the Laplacian, we obtain

$$\mathrm{Re}\left(\overline{z}_k \frac{\partial^2 z_k}{\partial \xi_j^2}\right)_{|\xi=\xi^*} = \mathrm{Re}\left(\mathrm{e}^{i\eta_k}\frac{\partial^2 z_k}{\partial \xi_j^2}\right)_{|\xi=\xi^*} =: I_j + II_j + III_j$$

with

$$I_j = \mathrm{Re}\left(\mathrm{e}^{i\eta_k}\frac{-\frac{\partial^2 P}{\partial \xi_j^2}}{\frac{\partial P}{\partial z}}\right) = (-1)^{j+k}\frac{D_{jk}}{D} \ ,$$

$$\begin{aligned}
II_j &= 2\mathrm{Re}\left(\mathrm{e}^{i\eta_k}\frac{\frac{\partial^2 P}{\partial\xi_j\partial z}\frac{\partial P}{\partial\xi_j}}{\left(\frac{\partial P}{\partial z}\right)^2}\right)\\
&= 2\mathrm{Re}\left(\frac{i\left(D_{jk}^2\left(i+\frac{1}{\xi_j^*-\eta_k}+i\sum_{l\neq k}a_{lk}\right)+\sum_{l\neq k}(-1)^{k+l}D_{jk}D_{jl}\left(\frac{1}{\xi_j^*-\eta_l}+ia_{lk}\right)\right)}{D^2}\right)\\
&= -\left((N-1)\frac{D_{jk}^2}{D^2}+\frac{\sum_{l=1}^{N-1}(-1)^{k+l}D_{jk}D_{jl}}{D^2}\right),
\end{aligned}$$

and

$$III_j = -\mathrm{Re}\left(\mathrm{e}^{i\eta_k}\frac{\left(\frac{\partial P}{\partial\xi_j}\right)^2\frac{\partial^2 P}{\partial z^2}}{\left(\frac{\partial P}{\partial z}\right)^3}\right) = \mathrm{Re}\left(\frac{2\sum_{l\neq k}a_{lk}D_{jk}^2}{D^2}\right) = (N-2)\frac{D_{jk}^2}{D^2}.$$

Remark that

$$\sum_{l=1}^{N-1}(-1)^{j+l}D_{jl} = D-(-1)^{j+N}\det\left(\frac{1}{\xi_r^*-\eta_l}\right)_{r\neq j}.$$

Putting these identities together, we obtain

$$\begin{aligned}
\sum_{j=1}^{N}\frac{\partial^2}{\partial\xi_j^2}\left(\frac{|z_k|^2}{2}\right)_{|\xi=\xi^*} &= \sum_{j=1}^{N}\left(\sum_{l=1}^{N-1}(-1)^{k+l+1}\frac{D_{jk}D_{jl}}{D^2}+(-1)^{j+k}\frac{D_{jk}}{D}\right)\\
&= (-1)^{k+N}\frac{1}{D^2}\sum_j D_{jk}\det\left(\frac{1}{\xi_r^*-\eta_l}\right)_{r\neq j}
\end{aligned}$$

It remains to check that the following analytic expression in η,

$$(-1)^{k+N}\sum_{j=1}^{N}D_{jk}\det\left(\frac{1}{\xi_r^*-\eta_l}\right)_{r\neq j}$$

is not identically zero. This follows from the fact that the limit as η_k tends to infinity of

$$(-1)^{k+N}\eta_k^2\sum_{j=1}^{N}D_{jk}\det\left(\frac{1}{\xi_r^*-\eta_l}\right)_{r\neq j}$$

equals

$$\sum_{j=1}^{N} \det \left(\left(\frac{1}{\xi_r^* - \eta_l} \right)_{r \neq j, l \neq k}, 1_N \right)^2$$

which is clearly not zero .
This completes the proof of Lemma 15, hence of the first part of Proposition 4. □

Remark 3. — *At this stage, let us notice that the above construction would break down if, instead of formula (6.2.2), we used for $u^{\delta,\varepsilon}$ the more natural choice*

$$u\left((\mathbf{s}, \delta(1+\varepsilon\xi_1), \delta(1+\varepsilon\eta_1), \dots, \delta(1+\varepsilon\xi_N), \delta(1+\varepsilon\eta_N)\right), (\mathbf{\Psi}, 1, \dots, 1)),$$

with real numbers $\xi_1 > \eta_1 > \cdots > \xi_N > \eta_N$. Indeed, in that case, the corresponding matrix $\mathscr{C}_\varepsilon(z)$ would be

$$\left(\frac{1 + \varepsilon\xi_j - z(1+\varepsilon\eta_k)\mathrm{e}^{-i(\xi_j - \eta_k) - i\varepsilon\frac{\xi_j^2 - \eta_k^2}{2}}}{\varepsilon(\xi_j - \eta_k)(2 + \varepsilon(\xi_j + \eta_k))} \right),$$

and, as ε tends to 0, its determinant would be equivalent to

$$\frac{1}{(2\varepsilon)^N} \det \left(\frac{1 - z\mathrm{e}^{-i(\xi_j - \eta_k)}}{\xi_j - \eta_k} \right)$$

The product of the N zeroes of the polynomial in z in the right hand side is clearly of modulus 1. Since, from Theorem 2, all these zeroes live outside the open unit disc, they have to be located on the unit circle, which is precisely the opposite of what we want.

6.4. The singular behavior

Let us come to the proof of the second part of Proposition 4. We are going to focus our analysis near the point $z = 1$, where the singularity of $u^{\delta,\varepsilon}(z)$ takes place. To accomplish this, we introduce a localized version of the H^s norm. Given $s \in \left(\frac{1}{2}, 1\right)$, it is classical (see e.g. [39]) that, for

every $u \in H^s_+(\mathbb{S}^1)$, the following equivalence of norms holds,

$$\left(\sum_{n=1}^{\infty} n^{2s}|\hat{u}(n)|^2\right)^{\frac{1}{2}} \simeq \left(\int\limits_{|z|<1} |u'(z)|^2(1-|z|^2)^{1-2s}\, dL(z)\right)^{\frac{1}{2}} ,$$

where L denotes the Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$ and u' denotes the holomorphic derivative of u.

Given a small constant $\theta > 0$ to be fixed later, we introduce the following subset of $\mathbb{D}$,

$$D_{\theta,\varepsilon} := \{z \in \mathbb{C} : |z| < 1\ ,\ |z-1| < \theta\varepsilon^{2(N-1)}\}\ ,$$

and the corresponding localized H^s norm,

$$\|u\|_{s,\theta,\varepsilon} := \left(\int\limits_{D_{\theta,\varepsilon}} |u'(z)|^2(1-|z|^2)^{1-2s}\, dL(z)\right)^{\frac{1}{2}} ,$$

so that we have the following inequality,

$$\forall u \in H^s_+\ ,\ \|u\|_{H^s} \geq C_s\|u\|_{s,\theta,\varepsilon}\ .$$

Notice that an elementary computation yields

$$\left(\int\limits_{D_{\theta,\varepsilon}} (1-|z|^2)^{1-2s}\, dL(z)\right)^{\frac{1}{2}} \simeq (\sqrt{\theta}\varepsilon^{N-1})^{3-2s}\ .$$

The second part of Proposition 4 therefore comes from the following

Proposition 5. — *There exists $\theta^* = \theta^*(\xi,\eta) > 0$ and a dense open subset $\mathcal{X}'''_N$ such that, for $\theta \leq \theta^*$, for every $(\xi,\eta) \in \mathcal{X}'''_N$,*

$$\forall z \in D_{\theta,\varepsilon}\ ,\ |(u^{\delta,\varepsilon})'(z)| \geq C_{\xi,\eta}\frac{\delta}{\varepsilon^{2(N-1)}}\ ,$$

for some $C_{\xi,\eta} > 0$, uniformly in δ, ε such that $\varepsilon \ll \delta \ll 1$.

Indeed, the proof of Proposition 4 is completed by denoting by $\mathcal{X}'_N$ the intersection of $\mathcal{X}'''_N$ and the nonempty open subset $\mathcal{X}''_N$ provided by the proof of the first part of the proposition.

Let us prove Proposition 5. Deriving with respect to z the formula (6.2.2) giving $u^{\delta,\varepsilon}$, we get

$$(u^{\delta,\varepsilon})'(z) = \left\langle \dot{\mathscr{C}}_{\delta,\varepsilon}\mathscr{C}_{\delta,\varepsilon}(z)^{-1}\begin{pmatrix}\mathbf{\Psi}_q\\ 1_N\end{pmatrix}, {}^t\mathscr{C}_{\delta,\varepsilon}(z)^{-1}\begin{pmatrix}1_q\\ 1_N\end{pmatrix}\right\rangle \tag{6.4.1}$$

where

$$\dot{\mathscr{C}}_{\delta,\varepsilon} = \begin{pmatrix}\dot{\mathscr{E}} & \dot{\mathscr{A}}_{\delta,\varepsilon}\\ \dot{\mathscr{B}}_{\delta,\varepsilon} & \frac{1}{\delta}\dot{\mathscr{C}}_\varepsilon\end{pmatrix} \tag{6.4.2}$$

with

$$\dot{\mathscr{E}} = \left(\frac{\sigma_b\Psi_{2a-1}\Psi_{2b}}{\rho_a^2-\sigma_b^2}\right)_{1\le a,b\le q}, \tag{6.4.3}$$

$$\dot{\mathscr{A}}_{\delta,\varepsilon} = \left(\left(\frac{\delta(1+\varepsilon\eta_k)\Psi_{2a-1}}{\rho_a^2-\delta^2(1+\varepsilon\eta_k)^2}\right)_{\substack{1\le a\le q\\ 1\le k\le N-1}}, \mathbf{0}_{1\le a\le q}\right) \tag{6.4.4}$$

$$\dot{\mathscr{B}}_{\delta,\varepsilon} = \left(\frac{\sigma_b\Psi_{2b}}{\delta^2(1+\varepsilon\xi_j)^2-\sigma_b^2}\right)_{\substack{1\le j\le N\\ 1\le b\le q}} \tag{6.4.5}$$

$$\dot{\mathscr{C}}_\varepsilon = (\dot{c}_{\varepsilon,jk})_{1\le j,k\le N}, \tag{6.4.6}$$

$$\dot{c}_{\varepsilon,jk} := \frac{1+\varepsilon\eta_k}{(1+\varepsilon\xi_j)^2-(1+\varepsilon\eta_k)^2}\,,\ 1\le k\le N-1\,,\ \dot{c}_{\varepsilon,jN} := 0\,,\ 1\le j\le N\,.$$

Let us introduce the following notation.

$$\mathscr{C}_{\delta,\varepsilon}(z)^{-1}\begin{pmatrix}\mathbf{\Psi_q}\\ 1_N\end{pmatrix} =: \begin{pmatrix}X_q^{\delta,\varepsilon}(z)\\ Y_N^{\delta,\varepsilon}(z)\end{pmatrix}$$

and

$${}^t\mathscr{C}_{\delta,\varepsilon}(z)^{-1}\begin{pmatrix}1_q\\ 1_N\end{pmatrix} =: \begin{pmatrix}\hat{X}_q^{\delta,\varepsilon}(z)\\ \hat{Y}_N^{\delta,\varepsilon}(z)\end{pmatrix}.$$

It gives rise to the following equations

$$\begin{cases}\mathscr{E}(z)X_q^{\delta,\varepsilon}(z)+\mathscr{A}_{\delta,\varepsilon}(z)Y_N^{\delta,\varepsilon}(z) &= \mathbf{\Psi_q}\\ \mathscr{B}_{\delta,\varepsilon}(z)X_q^{\delta,\varepsilon}(z)+\frac{1}{\delta}\mathscr{C}_\varepsilon(z)Y_N^{\delta,\varepsilon}(z) &= 1_N\end{cases}$$

$$\left\{\begin{array}{lcl} {}^t\mathscr{E}(z)\hat{X}_q^{\delta,\varepsilon}(z) +^t \mathscr{B}_{\delta,\varepsilon}(z)\hat{Y}_N^{\delta,\varepsilon}(z) & = & 1_q \\ {}^t\mathscr{A}_{\delta,\varepsilon}(z)\hat{X}_q^{\delta,\varepsilon}(z) + \frac{1}{\delta}{}^t\mathscr{C}_\varepsilon(z)\hat{Y}_N^{\delta,\varepsilon}(z) & = & 1_N \end{array}\right.$$

Hence, setting

$$\mathscr{J}_{\delta,\varepsilon}(z) := \mathscr{E}(z) - \delta\mathscr{A}_{\delta,\varepsilon}(z)\mathscr{C}_\varepsilon(z)^{-1}\mathscr{B}_{\delta,\varepsilon}(z)\ ,$$

we obtain

$$\mathscr{J}_{\delta,\varepsilon}(z)X_q^{\delta,\varepsilon}(z) = \mathbf{\Psi_q} - \delta\mathscr{A}_{\delta,\varepsilon}(z)\mathscr{C}_\varepsilon(z)^{-1}1_N \tag{6.4.7}$$

$${}^t\mathscr{J}_{\delta,\varepsilon}(z)\hat{X}_q^{\delta,\varepsilon}(z) = 1_q - \delta^t\mathscr{B}_{\delta,\varepsilon}(z)^t\mathscr{C}_\varepsilon(z)^{-1}1_N \tag{6.4.8}$$

$$Y_N^{\delta,\varepsilon}(z) = \delta\mathscr{C}_\varepsilon(z)^{-1}(1_N - \mathscr{B}_{\delta,\varepsilon}(z)X_q^{\delta,\varepsilon}(z)) \tag{6.4.9}$$

$$\hat{Y}_N^{\delta,\varepsilon}(z) = \delta^t\mathscr{C}_\varepsilon(z)^{-1}(1_N -^t \mathscr{A}_{\delta,\varepsilon}(z)\hat{X}_q^{\delta,\varepsilon}(z)). \tag{6.4.10}$$

Our main analysis lies in the following approximation result.

Proposition 6. — *For the norm $L^\infty(D_{\theta,\varepsilon})$, with $\theta < \theta^*(\xi,\eta)$ small enough, we have, uniformly in δ,ε such that $\varepsilon \ll \delta \ll 1$,*

$$\begin{array}{rcl} X_q^{\delta,\varepsilon}(z) & = & \mathscr{E}(1)^{-1}\mathbf{\Psi_q} + \mathcal{O}(\delta) \\ \hat{X}_q^{\delta,\varepsilon}(z) & = & {}^t\mathscr{E}(1)^{-1}1_q + \mathcal{O}(\delta) \\ Y_N^{\delta,\varepsilon}(z) & = & \alpha\delta\,\mathscr{C}_\varepsilon(z)^{-1}(1_N) + \mathcal{O}\left(\dfrac{\delta^2}{\varepsilon^{N-2}}\right) \\ \hat{Y}_N^{\delta,\varepsilon}(z) & = & \beta\delta\,{}^t\mathscr{C}_\varepsilon(z)^{-1}(1_N) + \mathcal{O}\left(\dfrac{\delta^2}{\varepsilon^{N-1}}\right) \end{array}$$

where

$$\begin{array}{rcl} \alpha : & = & 1 - \left\langle \mathscr{E}(1)^{-1}(\mathbf{\Psi_q}), \left(\dfrac{\Psi_{2b}}{\sigma_b}\right)_{1\le b\le q}\right\rangle, \\ \beta : & = & 1 - \left\langle {}^t\mathscr{E}(1)^{-1}(1_q), \left(\dfrac{1}{\rho_a}\right)_{1\le a\le q}\right\rangle\ . \end{array}$$

Proof. — The proof involves several steps. The first one consists in a number of cancellations in the action of the matrix $\mathscr{C}_\varepsilon(1)^{-1}$ on some special vectors. From the formula giving $\mathscr{C}_\varepsilon(z)$, we have

$$\mathscr{C}_\varepsilon(1) = \left(\left(\frac{1}{2+\varepsilon(\xi_j+\eta_k)} \right)_{1\le j\le N; 1\le k\le N-1}, \left(\frac{1}{1+\varepsilon\xi_j} \right)_{1\le j\le N} \right)$$

which is a Cauchy matrix. From Cramer's formulae and formula 6.3.1, the inverse of a Cauchy matrix is given by

$$\left(\left(\frac{1}{a_j+b_k} \right)^{-1} \right)_{kj} = \left((-1)^{j+k} \frac{\lambda_j \mu_k}{a_j+b_k} \right)_{kj} \tag{6.4.11}$$

with

$$\lambda_j = \frac{\prod_l (a_j+b_l)}{\prod_{i<j}(a_i-a_j)\prod_{r>j}(a_j-a_r)}$$

and

$$\mu_k = \frac{\prod_l (a_l+b_k)}{\prod_{i<k}(b_i-b_k)\prod_{r>k}(b_k-b_r)}.$$

Applying this formula to $\mathscr{C}_\varepsilon(1)$ with

$$a_j = 1+\varepsilon\xi_j,\ 1\le j\le N,\ b_k = 1+\varepsilon\eta_k,\ 1\le k\le N-1,\ b_N = 0,$$

we get

$$\begin{aligned} \left(\mathscr{C}_\varepsilon(1)^{-1}\right)_{kj} &= \frac{(-1)^{j+k}\lambda_j(\varepsilon)\mu_k(\varepsilon)}{2+\varepsilon(\xi_j+\eta_k)} \text{ if } k\le N-1; \\ \left(\mathscr{C}_\varepsilon(1)^{-1}\right)_{Nj} &= \frac{(-1)^{j+N}\lambda_j(\varepsilon)\mu_N(\varepsilon)}{1+\varepsilon\xi_j}, \end{aligned}$$

with

$$\lambda_j(\varepsilon) = \frac{2^{N-1}}{\varepsilon^{N-1}} \frac{(1+\varepsilon\xi_j)\prod_l(1+\varepsilon(\xi_j+\eta_l)/2)}{\xi'_j},\ 1\le j\le N,$$

and

$$\mu_k(\varepsilon) = \frac{2^N}{\varepsilon^{N-2}} \frac{\prod_l(1+\varepsilon(\xi_l+\eta_k)/2)}{\eta'_k(1+\varepsilon\eta_k)},\ k\le N-1,\ \mu_N(\varepsilon) = \frac{\prod_i(1+\varepsilon\xi_i)}{\prod_k(1+\varepsilon\eta_k)}$$

where we have set

$$\xi'_j := \prod_{i<j}(\xi_i-\xi_j)\prod_{r>j}(\xi_j-\xi_r),\ \eta'_k := \prod_{i<k}(\eta_i-\eta_k)\prod_{r>k}(\eta_k-\eta_r).$$

Finally, introduce the following special vectors $V_m(\varepsilon)$ and $W_p(\varepsilon)$ in $\mathbb{R}^N$.

$$\begin{aligned} V_m(\varepsilon)_j : &= (1+\varepsilon\xi_j)^m \ , \ m \geq 0 \ , \ j = 1, \dots, N, \\ W_p(\varepsilon)_k : &= (1+\varepsilon\eta_k)^p(1-\delta_{kN}) \ , \ p \geq 1 \ , \ W_0(\varepsilon)_k := 1, \ , \ k = 1, \dots, N, \end{aligned}$$

where δ_{kN} denotes the Kronecker symbol.

Lemma 16. — *The following identities hold for every $(\xi, \eta) \in \mathcal{X}_N$, uniformly in $\varepsilon \ll 1$.*

$$\big(\mathscr{C}_\varepsilon(1)^{-1}(1_N)\big)_k = (-1)^{k+1}\mu_k(\varepsilon) \ , \ 1 \leq k \leq N \ . \tag{6.4.12}$$

$$\big({}^t\mathscr{C}_\varepsilon(1)^{-1}(1_N)\big)_j = (-1)^{j+1}\lambda_j(\varepsilon) + \mathcal{O}\left(\frac{1}{\varepsilon^{N-2}}\right) \ , \ 1 \leq j \leq N \ . \tag{6.4.13}$$

Furthermore, for every $(\xi, \eta) \in \mathcal{X}_N$, there exist a constant $C_{\xi,\eta}$ and $\theta^*(\xi, \eta)$ such that, for the norm $L^\infty(D_{\theta,\varepsilon})$ with $\theta < \theta^*(\xi, \eta)$, uniformly for $0 < \varepsilon \leq 1$, we have the estimates

$$|\mathscr{C}_\varepsilon(z)^{-1}(V_m(\varepsilon))| \leq C_{\xi,\eta}\frac{(1+\xi_1)^m}{\varepsilon^{N-2}} \ , \ m \geq 0 \ . \tag{6.4.14}$$

$$|{}^t\mathscr{C}_\varepsilon(z)^{-1}(W_p(\varepsilon))| \leq C_{\xi,\eta}\frac{(1+\eta_1)^{p-1}}{\varepsilon^{N-1}} \ , \ p \geq 1 \ . \tag{6.4.15}$$

$$|\langle\mathscr{C}_\varepsilon(z)^{-1}(V_m(\varepsilon)), W_p(\varepsilon)\rangle| \leq C_{\xi,\eta}(1+\xi_1)^m(1+\eta_1)^p, m, p \geq 0 \ . \tag{6.4.16}$$

Proof. — The main algebraic cancellation is displayed in the following lemma.

Lemma 17. — *Let a, b with $0 \leq a+b \leq N-1$,*

$$\sum_{j=1}^N (-1)^j \frac{\xi_j^a}{\xi_j'} \sum_{|I|=b, I\subset\{1,\dots,N\}\setminus\{j\}} (\prod_{i\in I}\xi_i) = \begin{cases} 0 & \textit{if } 0 \leq a+b < N-1 \\ -1 & \textit{if } a = N-1, b = 0. \end{cases}$$

Analogous identities hold for the η's.

Proof. — We view

$$\sum_{j=1}^N (-1)^j \frac{\xi_j^a}{\xi_j'} \sum_{|I|=b, I\subset\{1,\dots,N\}\setminus\{j\}} (\prod_{i\in I}\xi_i)$$

as a rational function of ξ_N denoted by $Q_{a,b}(\xi_N)$. Its poles are simple and equal to $\xi_1, \dots, \xi_{N-1}$. Identifying the residue at each of these poles, we get

$$\begin{aligned}
&\mathrm{Res}(Q_{a,b}; \xi_N = \xi_r) = \\
&= \frac{(-1)^{r+1}\xi_r^a \left[\sum_{|I'|=b-1, I'\subset\{1,\dots,N-1\}\setminus\{r\}} (\prod_{i\in I'} \xi_i)\xi_r + \sum_{|I|=b, I\subset\{1,\dots,N-1\}\setminus\{r\}} (\prod_{i\in I} \xi_i)\right]}{\prod_{i<r}(\xi_i - \xi_r)\prod_{N-1\geq j>r}(\xi_r - \xi_j)} \\
&+ \frac{(-1)^{N+1}\xi_r^a \sum_{|I|=b, I\subset\{1,\dots,N-1\}} (\prod_{i\in I} \xi_i)}{\prod_{i<N, i\neq r}(\xi_i - \xi_r)} \\
&= 0 \ .
\end{aligned}$$

Hence $Q_{a,b}(\xi_N)$ is in fact a polynomial. If $a + b < N - 1$, it tends to 0 as ξ_N tends to ∞, therefore it is identically 0. If $a = N - 1$ and $b = 0$ it tends to -1. This completes the proof. □

Let us come to the proof of Lemma 16. From the formulae giving $\mathscr{C}_\varepsilon(1)^{-1}$, we get, if $k \leq N - 1$,

$$\begin{aligned}
\left(\mathscr{C}_\varepsilon(1)^{-1}(1_N)\right)_k &= (-1)^k \mu_k(\varepsilon) \sum_{j=1}^N \frac{(-1)^j \lambda_j(\varepsilon)}{2 + \varepsilon(\xi_j + \eta_k)} \\
&= (-1)^k \mu_k(\varepsilon) \frac{2^{N-2}}{\varepsilon^{N-1}} \sum_{j=1}^N (-1)^j \frac{(1 + \varepsilon\xi_j)}{\xi'_j} \prod_{l\neq k} \left(1 + \varepsilon \frac{\xi_j + \eta_l}{2}\right) ;
\end{aligned}$$

and

$$\begin{aligned}
\left((\mathscr{C}_\varepsilon(1)^{-1}(1_N)\right)_N &= (-1)^N \mu_N(\varepsilon) \sum_{j=1}^N \frac{(-1)^j \lambda_j(\varepsilon)}{1 + \varepsilon\xi_j} \\
&= (-1)^N \mu_N(\varepsilon) \frac{2^{N-1}}{\varepsilon^{N-1}} \sum_{j=1}^N \frac{(-1)^j}{\xi'_j} \prod_{l=1}^{N-1} \left(1 + \varepsilon \frac{\xi_j + \eta_l}{2}\right) .
\end{aligned}$$

Expanding in powers of ε the above formula giving $(\mathscr{C}_\varepsilon(1)^{-1}(1_N))_k$, and using Lemma 17 with $b = 0$, we infer

$$\left(\mathscr{C}_\varepsilon(1)^{-1}(1_N)\right)_k = (-1)^{k+1}\mu_k(\varepsilon) \ , \ \ k \leq N \ ,$$

which is (6.4.12). We now compute ${}^t\mathscr{C}_\varepsilon^{-1}(1)(1_N)$ in a similar way.

$$\begin{aligned}\left({}^t\mathscr{C}_\varepsilon(1)^{-1}(1_N)\right)_j &= (-1)^j\lambda_j(\varepsilon)\left(\sum_{k=1}^{N-1}\frac{(-1)^k\mu_k(\varepsilon)}{2+\varepsilon(\xi_j+\eta_k)}+\frac{(-1)^N\mu_N(\varepsilon)}{1+\varepsilon\xi_j}\right)\\ &=: (-1)^j\lambda_j(\varepsilon)\tilde{S}_j(\varepsilon).\end{aligned}$$

We expand in powers of ε and use again Lemma 17 with $b=0$, changing N into $N-1$. We get

$$\begin{aligned}\tilde{S}_j(\varepsilon) &= \frac{2^{N-1}}{\varepsilon^{N-2}}\sum_{k=1}^{N-1}\frac{(-1)^k\prod_{i\neq j}\left(1+\varepsilon\frac{\xi_i+\eta_k}{2}\right)}{\eta_k'(1+\varepsilon\eta_k)}+(-1)^N+\mathcal{O}(\varepsilon)\\ &= (-1)^N+\mathcal{O}(\varepsilon)+\frac{2^{N-1}}{\varepsilon^{N-2}}\sum_{k=1}^{N-1}(-1)^k\left(\varepsilon^{N-2}\sum_{p+q=N-2}(-1)^qC_{N-1}^p\frac{\eta_k^{p+q}}{2^p\eta_k'}+\mathcal{O}(\varepsilon^{N-1})\right)\\ &= -1+\mathcal{O}(\varepsilon).\end{aligned}$$

As a consequence, we infer

$$\left({}^t\mathscr{C}_\varepsilon(1)^{-1}(1_N)\right)_j=(-1)^{j+1}\lambda_j(\varepsilon)(1+\mathcal{O}(\varepsilon))\ ,$$

which is (6.4.13). We next prove (6.4.14) and (6.4.15), in the special case $z=1$. Notice that the cases $m=0$ and $p=0$ were addressed above, so we may assume $m,p\geq 1$. We have

$$(\mathscr{C}_\varepsilon(1)^{-1}(V_m(\varepsilon))_k=(-1)^k\mu_k(\varepsilon)\frac{2^{N-1}}{\varepsilon^{N-1}}h_{m,k}(\varepsilon,\xi,\eta),$$

with

$$h_{m,k}(\varepsilon,\xi,\eta):=\begin{cases}\displaystyle\frac{1}{2}\sum_{j=1}^N(-1)^j(1+\varepsilon\xi_j)^{m+1}\,\frac{\prod_{l\neq k}\left(1+\varepsilon\frac{\xi_j+\eta_l}{2}\right)}{\xi_j'} & \text{if } k\leq N-1\ ,\\[2ex] \displaystyle\sum_{j=1}^N(-1)^j(1+\varepsilon\xi_j)^m\,\frac{\prod_{l=1}^{N-1}\left(1+\varepsilon\frac{\xi_j+\eta_l}{2}\right)}{\xi_j'} & \text{if } k=N\ .\end{cases}$$

Notice that $h_{m,k}$ is a holomorphic function of ε. From Lemma 17 with $b=0$, the derivatives $\partial_\varepsilon^r h_{m,k}(0,\xi,\eta)$ vanish for $r<N-1$. Therefore, from the maximum principle,

$$\text{(6.4.17)}\quad \frac{|h_{m,k}(\varepsilon,\xi,\eta)|}{\varepsilon^{N-1}}\leq\sup_{|\zeta|=1}|h_{m,k}(\zeta,\xi,\eta)|\leq C_{\xi,\eta}(1+\xi_1)^m\ ,\ 0<\varepsilon\leq 1\ .$$

This provides estimate (6.4.14) for $z = 1$. Estimate (6.4.15) for $z = 1$ is obtained in the same way, by observing that

$$ {}^t\mathscr{C}_\varepsilon(1)^{-1}(W_p(\varepsilon)) = (-1)^j \lambda_j(\varepsilon)\frac{2^N}{\varepsilon^{N-2}} g_{p,j}(\varepsilon,\xi,\eta),\ p \geq 1 \ , $$

where we have set

$$ \begin{aligned} g_{p,j}(\varepsilon,\xi,\eta) &:= \frac{\varepsilon^{N-2}}{2^N}\sum_{k=1}^{N-1}(-1)^k \frac{\mu_k(\varepsilon)(1+\varepsilon\eta_k)^p}{2+\varepsilon(\xi_j+\eta_k)} \\ &= \frac{1}{2}\sum_{k=1}^{N-1}(-1)^k \frac{\prod_{l\neq j}(1+\varepsilon(\xi_l+\eta_k)/2)}{\eta'_k}(1+\varepsilon\eta_k)^{p-1} \ , \end{aligned} $$

and proving in the same way that

$$ \frac{|g_{p,j}(\varepsilon,\xi,\eta)|}{\varepsilon^{N-2}} \leq \sup_{|\zeta|=1}|g_{p,j}(\zeta,\xi,\eta)| \leq C_{\xi,\eta}(1+\eta_1)^{p-1}. $$

As for (6.4.16) for $z = 1$, we compute

$$ \langle C_\varepsilon(1)^{-1}(V_m(\varepsilon)), W_p(\varepsilon)\rangle = \frac{2^{2N-1}}{\varepsilon^{2N-3}} f_{pm}(\varepsilon,\xi,\eta) \ , $$

where

$$ \begin{aligned} & f_{0m}(\varepsilon,\xi,\eta) = \\ = \ & \frac{1}{2}\sum_{1\leq j\leq N, 1\leq k\leq N-1}(-1)^{j+k}\frac{(1+\varepsilon\xi_j)^{m+1}\prod_{l\neq k}(1+\varepsilon\frac{\xi_j+\eta_l}{2})\prod_i(1+\varepsilon\frac{\xi_i+\eta_k}{2})}{(1+\varepsilon\eta_k)\xi'_j\eta'_k} \\ & + \frac{\varepsilon^{N-2}}{2^N}\frac{\prod_{i=1}^N(1+\varepsilon\xi_i)}{\prod_{l=1}^{N-1}(1+\varepsilon\eta_l)}\sum_{j=1}^N(-1)^{j+N}\frac{(1+\varepsilon\xi_j)^m\prod_l(1+\varepsilon\frac{\xi_j+\eta_l}{2})}{\xi'_j} \\ & =: \tilde{f}_{0m}(\varepsilon,\xi,\eta) + (-1)^N\frac{\varepsilon^{N-2}}{2^N}\frac{\prod_{i=1}^N(1+\varepsilon\xi_i)}{\prod_{l=1}^{N-1}(1+\varepsilon\eta_l)}h_{mN}(\varepsilon,\xi,\eta) \ , \end{aligned} $$

and, for $p \geq 1$,

$$ \begin{aligned} & f_{pm}(\varepsilon,\xi,\eta) = \\ = \ & \frac{1}{2}\sum_{1\leq j\leq N, 1\leq k\leq N-1}(-1)^{j+k}\frac{(1+\varepsilon\xi_j)^{m+1}(1+\varepsilon\eta_k)^{p-1}\prod_{l\neq k}(1+\varepsilon\frac{\xi_j+\eta_l}{2})\prod_i(1+\varepsilon\frac{\xi_i+\eta_k}{2})}{\xi'_j\eta'_k} \ . \end{aligned} $$

Since we already proved in (6.4.17) that

$$ |h_{mN}(\varepsilon,\xi,\eta)| \leq C_{\xi\eta}\varepsilon^{N-1}(1+\xi_1)^m \ , $$

estimate (6.4.16) will be a consequence of

$$|\tilde f_{0m}(\varepsilon,\xi\eta)| \le C_{\xi,\eta}\varepsilon^{2N-3}(1+\xi_1)^m\ ,\ |\tilde f_{pm}(\varepsilon,\xi,\eta)| \le C_{\xi,\eta}\varepsilon^{2N-3}(1+\xi_1)^m(1+\eta_1)^{p-1}$$

for $m \ge 0, p \ge 1$. Notice that $\tilde f_{0m}(\zeta,\xi\eta)$ and $f_{pm}(\zeta,\xi,\eta)$ are holomorphic function of ζ for $|\zeta| < \varepsilon^*$. Using again the maximum principle, the proof of the lemma will be complete if we prove that for any $0 \le r < 2N-3$, all the derivatives $\partial_\varepsilon^r f_{pm}(0,\xi,\eta)$ and $\partial_\varepsilon^r \tilde f_{0m}(0,\xi,\eta)$ vanish.
Indeed, such a derivative involves a sum of terms

$$\sum_{1\le j\le N, 1\le k\le N-1} \frac{(-1)^{j+k}\xi_j^a\eta_k^t}{\xi_j'\eta_k'} \sum_{\substack{|L|=c,|I|=d\\ k\notin L}} \prod_{l\in L}(\xi_j+\eta_l)\prod_{i\in I}(\xi_i+\eta_k)$$

with $a+t+c+d = r \le 2N-4$. We symmetrize this expression by writing

$$\sum_{|I|=d}\prod_{i\in I}(\xi_i+\eta_k) = (\xi_j+\eta_k)\sum_{\substack{|I|=d-1\\ j\notin I}}\prod_{i\in I}(\xi_i+\eta_k) + \sum_{\substack{|I|=d\\ j\notin I}}\prod_{i\in I}(\xi_i+\eta_k)$$

which gives rise to a sum of terms

$$\sum_{1\le j\le N, 1\le k\le N-1} \frac{(-1)^{j+k}\xi_j^a\eta_k^t}{\xi_j'\eta_k'} \sum_{\substack{|L|=c,|I|=d\\ k\notin L, j\notin I}} \prod_{l\in L}(\xi_j+\eta_l)\prod_{i\in I}(\xi_i+\eta_k)$$

with $a+t+c+d = r \le 2N-4$. Expanding everything, we have to calculate

$$\sum_{1\le j\le N, 1\le k\le N-1} \frac{(-1)^{j+k}\xi_j^{a+s}\eta_k^{t+u}}{\xi_j'\eta_k'} \sum_{\substack{|L|=c-s,|I|=d-u\\ k\notin L, j\notin I}} \prod_{l\in L}\eta_l\prod_{i\in I}\xi_i$$

with either $a+s+d-u < N-1$ or $t+u+c-s < N-2$ since $(a+s+d-u)+(r+u+c-s) = a+t+c+d \le 2N-4$. Using Lemma 17, we get that all these terms vanish.

Finally, we show how to pass from estimates (6.4.14), (6.4.15), (6.4.16) for $z=1$ to uniform estimates for $z \in D_{\theta,\varepsilon}$. We write

$$\begin{aligned}
\mathscr{C}_\varepsilon(z) &= \mathscr{C}_\varepsilon(1) + (1-z)\dot{\mathscr{C}}_\varepsilon \\
&= \mathscr{C}_\varepsilon(1)(I + (1-z)\mathscr{C}_\varepsilon(1)^{-1}\dot{\mathscr{C}}_\varepsilon) \\
&= (I + (1-z)\dot{\mathscr{C}}_\varepsilon\mathscr{C}_\varepsilon(1)^{-1})\mathscr{C}_\varepsilon(1)\ .
\end{aligned}$$

In view of the expressions of $\dot{\mathscr{C}}_\varepsilon$ and of $\mathscr{C}_\varepsilon(1)^{-1}$, we have

$$|\mathscr{C}_\varepsilon(1)^{-1}\dot{\mathscr{C}}_\varepsilon| + |\dot{\mathscr{C}}_\varepsilon\mathscr{C}_\varepsilon(1)^{-1}| \leq \frac{A(\xi,\eta)}{\varepsilon^{2(N-1)}}\ .$$

Consequently, for $z \in D_{\theta,\varepsilon}$, we have

$$|1-z|(|\mathscr{C}_\varepsilon(1)^{-1}\dot{\mathscr{C}}_\varepsilon| + |\dot{\mathscr{C}}_\varepsilon\mathscr{C}_\varepsilon(1)^{-1}|) \leq A(\xi,\eta)\theta \leq \frac{1}{2}$$

if $\theta \leq \theta^*(\xi,\eta)$, and the matrices

$$S_\varepsilon(z) := I + (1-z)\mathscr{C}_\varepsilon(1)^{-1}\dot{\mathscr{C}}_\varepsilon\ ,\ \hat{S}_\varepsilon(z) := I + (1-z)\dot{\mathscr{C}}_\varepsilon\mathscr{C}_\varepsilon(1)^{-1}$$

are invertible, with

$$|S_\varepsilon(z)^{-1}| + |\hat{S}_\varepsilon(z)^{-1}| \leq 2$$

Estimates (6.4.14), (6.4.15), (6.4.16) on $D_{\theta,\varepsilon}$ are then consequences of the estimates for $z=1$ and of the identities

$$\begin{aligned}
\mathscr{C}_\varepsilon(z)^{-1}V_m(\varepsilon) &= S_\varepsilon(z)^{-1}\mathscr{C}_\varepsilon(1)^{-1}V_m(\varepsilon)\ ,\\
{}^t\mathscr{C}_\varepsilon(z)^{-1}W_p(\varepsilon) &= \hat{S}_\varepsilon(z)^{-1}\,{}^t\mathscr{C}_\varepsilon(1)^{-1}W_p(\varepsilon)\ ,
\end{aligned}$$

and

$$\begin{aligned}
&\langle\mathscr{C}_\varepsilon(z)^{-1}V_m(\varepsilon), W_p(\varepsilon)\rangle = \langle S_\varepsilon(z)^{-1}\mathscr{C}_\varepsilon(1)^{-1}V_m(\varepsilon), W_p(\varepsilon)\rangle\\
= \ &\langle\mathscr{C}_\varepsilon(1)^{-1}V_m(\varepsilon), W_p(\varepsilon)\rangle - (1-z)\langle\dot{\mathscr{C}}_\varepsilon\mathscr{C}_\varepsilon(1)^{-1}S_\varepsilon(z)^{-1}\mathscr{C}_\varepsilon(1)^{-1}V_m(\varepsilon), {}^t\mathscr{C}_\varepsilon(1)^{-1}W_p(\varepsilon)\rangle\ .
\end{aligned}$$

This completes the proof of Lemma 16. $\square$

As a consequence of Lemma 16, we get

Lemma 18. — *For $\theta < \theta^*(\xi,\eta)$, the following matrix expansions hold in $L^\infty(D_{\theta,\varepsilon})$ as ε, δ tend to 0.*

$$\begin{aligned}
\mathscr{A}_{\delta,\varepsilon}(z)\mathscr{C}_\varepsilon(z)^{-1} &= \left(\frac{1}{\rho_a}\right)_{1\leq a\leq q} \otimes^t \mathscr{C}_\varepsilon(z)^{-1}(1_N) + \mathcal{O}\left(\frac{\delta}{\varepsilon^{N-1}}\right)\\
\mathscr{C}_\varepsilon(z)^{-1}\mathscr{B}_{\delta,\varepsilon}(z) &= \mathscr{C}_\varepsilon(z)^{-1}(1_N) \otimes \left(\frac{\Psi_{2b}}{\sigma_b}\right)_{1\leq b\leq q} + \mathcal{O}\left(\frac{\delta}{\varepsilon^{N-2}}\right)
\end{aligned}$$

Furthermore, for $z \in D_{\theta,\varepsilon}$, the vectors

$$\mathscr{A}_{\delta,\varepsilon}(z)\mathscr{C}_\varepsilon(z)^{-1}(1_N),\ {}^t\mathscr{B}_{\delta,\varepsilon}(z)^t\mathscr{C}_\varepsilon(z)^{-1}(1_N)$$

and the matrix

$$\mathscr{A}_{\delta,\varepsilon}(z)\mathscr{C}_\varepsilon(z)^{-1}\mathscr{B}_{\delta,\varepsilon}(z)$$

are uniformly bounded as ε, δ tend to 0.

Proof. — Recall that

$$\begin{aligned}
\mathscr{A}_{\delta,\varepsilon}(z) &= \left(\left(\frac{\rho_a - \delta z(1+\varepsilon\eta_k)\Psi_{2a-1}}{\rho_a^2 - \delta^2(1+\varepsilon\eta_k)^2} \right)_{\substack{1\le a\le q\\ 1\le k\le N-1}}, \left(\frac{1}{\rho_a}\right)_{1\le a\le q} \right)\\
\mathscr{B}_{\delta,\varepsilon}(z) &= \left(\frac{\delta(1+\varepsilon\xi_j) - \sigma_b z\Psi_{2b}}{\delta^2(1+\varepsilon\xi_j)^2 - \sigma_b^2} \right)_{\substack{1\le j\le N\\ 1\le b\le q}}.
\end{aligned}$$

Expanding in powers of δ, we obtain

$$\begin{aligned}
\mathscr{A}_{\delta,\varepsilon}(z) &= \sum_{p=0}^{\infty} \delta^p \mathscr{A}_p(\varepsilon,z)\ ,\ \mathscr{A}_p(\varepsilon,z) := U_p(z)\otimes W_p(\varepsilon)\ ,\ p\ge 0,\\
\mathscr{B}_{\delta,\varepsilon}(z) &= \sum_{m=0}^{\infty} \delta^m \mathscr{B}_m(\varepsilon,z)\ ,\ \mathscr{B}_m(\varepsilon,z) := V_m(\varepsilon)\otimes T_m(z)\ ,\ m\ge 0,
\end{aligned}$$

where we have set

$$\begin{aligned}
U_p(z) &:= \left(\begin{cases} \dfrac{1}{\rho_a^{p+1}} \text{ if } p \text{ is even,} \\[2ex] -\dfrac{z\Psi_{2a-1}}{\rho_a^{p+1}} \text{ if } p \text{ is odd.} \end{cases} \right)_{1\le a\le q}\\
T_m(z) &:= \left(\begin{cases} \dfrac{z\Psi_{2b}}{\sigma_b^{m+1}} \text{ if } m \text{ is even,} \\ -\dfrac{1}{\sigma_b^{m+1}} \text{ if } m \text{ is odd.} \end{cases} \right)_{1\le b\le q}
\end{aligned}$$

Using these formulae, we get

$$\begin{aligned}
\mathscr{A}_{\delta,\varepsilon}(z)\mathscr{C}_\varepsilon(z)^{-1} &= \sum_{p=0}^{\infty} \delta^p U_p(z) \otimes^t \mathscr{C}_\varepsilon(z)^{-1} W_p(\varepsilon), \\
\mathscr{C}_\varepsilon(z)^{-1}\mathscr{B}_{\delta,\varepsilon}(z) &= \sum_{m=0}^{\infty} \delta^m \mathscr{C}_\varepsilon(z)^{-1} V_m(\varepsilon) \otimes T_m(z)\ , \\
\mathscr{A}_{\delta,\varepsilon}(z)\mathscr{C}_\varepsilon(z)^{-1}(1_N) &= \sum_{p=0}^{\infty} \delta^p \langle W_p(\varepsilon), \mathscr{C}_\varepsilon(z)^{-1}(1_N)\rangle U_p(z)\ , \\
{}^t\mathscr{B}_{\delta,\varepsilon}(z)^t\mathscr{C}_\varepsilon(z)^{-1}(1_N) &= \sum_{m=0}^{\infty} \delta^m \langle \mathscr{C}_\varepsilon(z)^{-1} V_m(\varepsilon), 1_N\rangle T_m(z)\ , \\
\mathscr{A}_{\delta,\varepsilon}(z)\mathscr{C}_\varepsilon(z)^{-1}\mathscr{B}_{\delta,\varepsilon}(z) &= \sum_{p,m\geq 0} \langle \mathscr{C}_\varepsilon(z)^{-1} V_m(\varepsilon), W_p(\varepsilon)\rangle U_p(z) \otimes T_m(z)\ .
\end{aligned}$$

The statement is then a direct consequence of Lemma 16. □

Let us complete the proof of Proposition 6. In view of Lemma 18, and of formulae (6.4.7), (6.4.8), (6.4.9), (6.4.10), we observe that

$$\mathscr{J}_{\delta,\varepsilon}(z) = \mathscr{E}(z) - \delta \mathscr{A}_{\delta,\varepsilon}(z)\mathscr{C}_\varepsilon(z)^{-1}\mathscr{B}_{\delta,\varepsilon}(z) = \mathscr{E}(z) + \mathcal{O}(\delta)$$

is invertible for $z \in D_{\theta,\varepsilon}$ and δ and ε small enough, and we get the formulae

$$\begin{aligned}
X_q^{\delta,\varepsilon}(z) &= \mathscr{J}_{\delta,\varepsilon}(z)^{-1}(\mathbf{\Psi_q} - \delta \mathscr{A}_{\delta,\varepsilon}(z)\mathscr{C}_\varepsilon(z)^{-1}1_N) \\
\hat{X}_q^{\delta,\varepsilon}(z) &= {}^t\mathscr{J}_{\delta,\varepsilon}(z)^{-1}(1_q - \delta^t\mathscr{B}_{\delta,\varepsilon}(z)\, {}^t\mathscr{C}_\varepsilon(z)^{-1}1_N) \\
Y_N^{\delta,\varepsilon}(z) &= \delta \mathscr{C}_\varepsilon(z)^{-1}(1_N - \mathscr{B}_{\delta,\varepsilon}(z) X_q^{\delta,\varepsilon}(z)) \\
\hat{Y}_N^{\delta,\varepsilon}(z) &= \delta\, {}^t\mathscr{C}_\varepsilon(z)^{-1}(1_N -{}^t \mathscr{A}_{\delta,\varepsilon}(z)\hat{X}_q^{\delta,\varepsilon}(z)).
\end{aligned}$$

Furthermore, observing that $\mathscr{E}(z)$ is invertible for z in a fixed neighborhood of $\mathbb{D}$, we have, if $z \in D_{\theta,\varepsilon}$,

$$\mathscr{E}(z)^{-1} = \mathscr{E}(1)^{-1} + O(\varepsilon^{2(N-1)})\ .$$

Applying again Lemma Lemma 18, this completes the proof. □

Let us complete the proof of Proposition 4. Proposition 6 and the expression of $(u^{\delta,\varepsilon})'(1)$ lead to

$$\begin{aligned}(u^{\delta,\varepsilon})'(z) &= \left\langle \dot{\mathscr{C}}_{\delta,\varepsilon}\begin{pmatrix} X_q^{\delta,\varepsilon}(z)\\ Y_N^{\delta,\varepsilon}(z)\end{pmatrix}, \begin{pmatrix} \hat{X}_q^{\delta,\varepsilon}(z)\\ \hat{Y}_N^{\delta,\varepsilon}(z)\end{pmatrix}\right\rangle \\ &= \left\langle \dot{\mathscr{C}}_{\delta,\varepsilon}\begin{pmatrix} \mathscr{E}(z)^{-1}\mathbf{\Psi_q}+\mathcal{O}(\delta)\\ \alpha\delta\mathscr{C}_\varepsilon(z)^{-1}(1_N)+\mathcal{O}\left(\frac{\delta^2}{\varepsilon^{N-2}}\right)\end{pmatrix}, \begin{pmatrix} {}^t\mathscr{E}(z)^{-1}1_q+\mathcal{O}(\delta)\\ \beta\delta^t\mathscr{C}_\varepsilon(z)^{-1}(1_N)+\mathcal{O}\left(\frac{\delta^2}{\varepsilon^{N-1}}\right)\end{pmatrix}\right\rangle .\end{aligned}$$

Observing that

$$\dot{\mathscr{C}}_{\delta,\varepsilon} = \begin{pmatrix}\mathcal{O}(1) & \mathcal{O}(\delta)\\ \mathcal{O}(1) & \frac{1}{\delta}\dot{\mathscr{C}}_\varepsilon\end{pmatrix} = \begin{pmatrix}\mathcal{O}(1) & \mathcal{O}(\delta)\\ \mathcal{O}(1) & \mathcal{O}\left(\frac{1}{\delta\varepsilon}\right)\end{pmatrix} ,$$

we infer, if $\varepsilon \ll \delta$,

$$\begin{aligned}(u^{\delta,\varepsilon})'(z) &= \left\langle \begin{pmatrix}\mathcal{O}(1)+\mathcal{O}\left(\frac{\delta^2}{\varepsilon^{N-2}}\right)\\ \mathcal{O}(1)+\alpha\dot{\mathscr{C}}_\varepsilon\mathscr{C}_\varepsilon(z)^{-1}(1_N)+\mathcal{O}\left(\frac{\delta}{\varepsilon^{N-1}}\right)\end{pmatrix}, \begin{pmatrix}\mathcal{O}(1)\\ \beta\delta^t\mathscr{C}_\varepsilon(z)^{-1}(1_N)+\mathcal{O}\left(\frac{\delta^2}{\varepsilon^{N-1}}\right)\end{pmatrix}\right\rangle \\ &= \alpha\beta\delta\langle\dot{\mathscr{C}}_\varepsilon\mathscr{C}_\varepsilon(z)^{-1}(1_N),{}^t\mathscr{C}_\varepsilon(z)^{-1}(1_N)\rangle+\mathcal{O}\left(\frac{\delta^2}{\varepsilon^{2(N-1)}}\right).\end{aligned}$$

Furthermore, writing again

$\mathscr{C}_\varepsilon(z)^{-1} = S_\varepsilon(z)^{-1}\mathscr{C}_\varepsilon(1)^{-1} = \mathscr{C}_\varepsilon(1)^{-1}-(1-z)\dot{\mathscr{C}}_\varepsilon\mathscr{C}_\varepsilon(1)^{-1}S_\varepsilon(z)^{-1}\mathscr{C}_\varepsilon(1)^{-1}$,

we infer

$$(u^{\delta,\varepsilon})'(z) = \alpha\beta\delta\langle\dot{\mathscr{C}}_\varepsilon\mathscr{C}_\varepsilon(1)^{-1}(1_N),{}^t\mathscr{C}_\varepsilon(1)^{-1}(1_N)\rangle+\mathcal{O}\left(\frac{\delta^2+\theta}{\varepsilon^{2(N-1)}}\right) .$$

We claim that the product

$$\alpha\beta = \left(1-\left\langle\mathscr{E}^{-1}\mathbf{\Psi_q},\left(\frac{\Psi_{2b}}{\sigma_b}\right)\right\rangle\right)\left(1-\left\langle{}^t\mathscr{E}^{-1}1_q,\left(\frac{1}{\rho_a}\right)\right\rangle\right)$$

is not zero. Indeed, write $Z_b := \mathscr{E}^{-1}\mathbf{\Psi_q}$ so that, for $a=1,\ldots,q$,

$$\sum_{b=1}^{q}\frac{\rho_a-\sigma_b\Psi_{2a-1}\Psi_{2b}}{\rho_a^2-\sigma_b^2}Z_b = \Psi_{2a-1}$$

or

$$\sum_{b=1}^{q}\frac{\rho_a\Psi_{2a-1}^{-1}-\sigma_b\Psi_{2b}}{\rho_a^2-\sigma_b^2}Z_b = 1 \ , \ a=1,\ldots,q. \tag{6.4.18}$$

Assume $\alpha = 0$, namely

$$\sum_{b=1}^{q} Z_b \frac{\Psi_{2b}}{\sigma_b} = 1 \ . \tag{6.4.19}$$

Then substracting (6.4.18) from (6.4.19) leads to

$$\rho_a \sum_{b=1}^{q} \frac{\rho_a - \sigma_b \Psi_{2a-1}^{-1} \Psi_{2b}^{-1}}{\rho_a^2 - \sigma_b^2} \frac{\Psi_{2b} Z_b}{\sigma_b} = 0 \ , \ a = 1, \ldots, q.$$

This is a contradiction since, from Theorem 2, the matrix

$$\left(\frac{\rho_a - \sigma_b \Psi_{2a-1}^{-1} \Psi_{2b}^{-1}}{\rho_a^2 - \sigma_b^2} \right)_{1 \le a,b \le q}$$

is known to be invertible since Ψ_{2a-1}^{-1} and Ψ_{2b}^{-1} are complex numbers of modulus 1. We conclude that $\alpha \neq 0$. A similar argument leads to $\beta \neq 0$.

Eventually, we calculate

$$\begin{aligned} \langle \dot{\mathscr{C}}_\varepsilon \mathscr{C}_\varepsilon^{-1}(1_N), {}^t\mathscr{C}_\varepsilon^{-1}(1_N) \rangle &= \sum_{\substack{1 \le j \le N \\ 1 \le k \le N-1}} \frac{(-1)^{j+k} \lambda_j(\varepsilon) \mu_k(\varepsilon)(1 + \varepsilon \eta_k)}{(1 + \varepsilon \xi_j)^2 - (1 + \varepsilon \eta_k)^2} (1 + \mathcal{O}(\varepsilon)) \\ &= \frac{2^{2(N-1)}}{\varepsilon^{2(N-1)}} \left(\sum_{1 \le j \le N} \sum_{1 \le k \le N-1} \frac{(-1)^{j+k}}{(\xi_j - \eta_k) \xi_j' \eta_k'} + \mathcal{O}(\varepsilon) \right) . \end{aligned}$$

Considering the analytic expression

$$\sum_{1 \le j \le N} \sum_{1 \le k \le N-1} \frac{(-1)^{j+k}}{(\xi_j - \eta_k) \xi_j' \eta_k'}$$

as a function of ξ_N, the pole η_{N-1} appears only once. Therefore this quantity does not vanish if (ξ, η) belongs to an open dense set $\mathcal{X}_N'''$ of V. In that case, if $\theta \le \theta^*(\xi, \eta)$ small enough, and $\varepsilon \ll \delta \ll 1$, we conclude

$$\forall z \in D_{\theta,\varepsilon} \ , \ |(u^{\delta,\varepsilon})'(z)| \ge \frac{C\delta}{\varepsilon^{2(N-1)}} \ .$$

This completes the proof of Proposition 5.

□

CHAPTER 7

GEOMETRY OF THE FOURIER TRANSFORM

This chapter is devoted to the proof of two results. The first one describes the restriction of the symplectic form to the finite dimensional manifolds made of symbols corresponding to pairs of Hankel operators with a given finite list of multiplicities. These manifolds turn out to involutive, and are the disjoint union of symplectic manifolds on which the nonlinear Fourier transform defines action angle variables for the cubic Szegő flow. The proof of this result uses the evolution of the nonlinear Fourier transform through the flows of the Szegő hierarchy introduced in [11] and used in [14]. The second result characterizes the sets of symbols associated to pairs of Hankel operators with the same singular values and Blaschke products admitting a given set of zeroes, as classes of some unitary equivalence. These sets are precisely tori obtained in the above symplectic manifolds by fixing the action variables and making angles vary.

7.1. Evolution under the Szegő hierarchy

The Szegő hierarchy was introduced in [11] and used in [12] and [14]. In [12], it was used to identify the symplectic form on the generic part of $\mathcal{V}(d)$. Similarly, our purpose in this section is to establish preliminary formulae, towards the identification of the symplectic form on $\mathcal{V}_{(d_1,\dots,d_n)}$ in section 7.2.

For the convenience of the reader, we recall the main properties of the hierarchy. For $y > 0$ and $u \in H^{\frac{1}{2}}_+$, we set

$$J^y(u) = ((I + yH_u^2)^{-1}(1)|1) \ .$$

Notice that the connection with the Szegő equation is made by

$$E(u) = \frac{1}{4}(\partial_y^2 J^y_{|y=0} - (\partial_y J^y_{|y=0})^2) \ .$$

Thanks to formula (3.2.7), $J^y(u)$ is a function of the singular values $s_r(u)$. For every $s > \frac{1}{2}$, J^y is a smooth real valued function on H^s_+, and its Hamiltonian vector field is given by

$$X_{J^y}(u) = 2iyw^y H_u w^y \ , \ w^y := (I + yH_u^2)^{-1}(1) \ ,$$

which is a Lipschitz vector field on bounded subsets of H^s_+. By the Cauchy–Lipschitz theorem, the evolution equation

$$\dot u = X_{J^y}(u) \tag{7.1.1}$$

admits local in time solutions for every initial data in H^s_+ for $s > 1$, and the lifetime is bounded from below if the data are bounded in H^s_+. We recall that this evolution equation admits a Lax pair structure ([14]).

Theorem 10. — *For every $u \in H^s_+$, we have*

$$\begin{aligned} H_{iX_{J^y}(u)} &= H_u F_u^y + F_u^y H_u \ , \\ K_{iX_{J^y}(u)} &= K_u G_u^y + G_u^y K_u \ , \\ G_u^y(h) &:= -yw^y \,\Pi(\overline{w^y}\, h) + y^2 H_u w^y \,\Pi(\overline{H_u w^y}\, h) \ , \\ F_u^y(h) &:= G_u^y(h) - y^2 (h|H_u w^y) H_u w^y \ . \end{aligned}$$

If $u \in C^\infty(\mathcal{I}, H^s_+)$ is a solution of equation (7.1.1) on a time interval $\mathcal{I}$, then

$$\begin{aligned} \frac{dH_u}{dt} &= [B_u^y, H_u] \ , \ \frac{dK_u}{dt} = [C_u^y, K_u] \ , \\ B_u^y &= -iF_u^y \ , \ C_u^y = -iG_u^y \ . \end{aligned}$$

In particular, $\Sigma_H(u_0) = \Sigma_H(u(t))$ and $\Sigma_K(u_0) = \Sigma_K(u(t))$ for every t, therefore $J^y(u(t))$ is a constant J^y. We now state the main result of this section.

Theorem 11. — *Let $u_0 \in H^s_+$, $s > 1$, with*

$$\Phi(u_0) = ((s_r), (\Psi_r)).$$

The solution of

$$\dot{u} = X_{J^y}(u) \ , \ u(0) = u_0 \ ,$$

is characterized by

$$\Phi(u(t)) = ((s_r), (\mathrm{e}^{i\omega_r t}\Psi_r)) \ , \ \omega_r := (-1)^{r-1}\frac{2yJ^y}{1+ys_r^2} \ .$$

Proof. — Let $\rho \in \Sigma_H(u_0)$. Denote by u_ρ the orthogonal projection of u on $E_u(\rho)$. Hence, $u_\rho = 1\!\!1_{\{\rho^2\}}(H_u^2)(u)$. Let us differentiate this equation with respect to time. We get from the Lax pair structure

$$\begin{aligned}\frac{du_\rho}{dt} &= [B_u^y, 1\!\!1_{\{\rho^2\}}(H_u^2)](u) + 1\!\!1_{\{\rho^2\}}(H_u^2)[B_u^y, H_u](1)\\ &= B_u^y(u_\rho) - 1\!\!1_{\{\rho^2\}}(H_u^2)(H_u(B_u^y(1))) \ .\end{aligned}$$

Since $B_u^y(1) = iyJ^yw^y$, and since $1\!\!1_{\{\rho^2\}}(H_u^2)(H_uw^y) = \frac{1}{1+y\rho^2}u_\rho$, we get

$$\frac{du_\rho}{dt} = B_u^y(u_\rho) + i\frac{yJ^y}{1+y\rho^2}u_\rho \ . \tag{7.1.2}$$

On the other hand, differentiating the equation

$$\rho u_\rho = \Psi H_u(u_\rho)$$

one obtains

$$\rho\frac{du_\rho}{dt} = \dot{\Psi}H_u(u_\rho) + \Psi\left([B_u^y, H_u](u_\rho) + H_u\left(\frac{du_\rho}{dt}\right)\right)$$

Hence, using the expression (7.1.2), we get

$$\rho\left(B_u^y(u_\rho) + i\frac{yJ^y}{1+y\rho^2}u_\rho\right) = \left(\dot{\Psi} - i\frac{yJ^y}{1+y\rho^2}\Psi\right)H_u(u_\rho) + \Psi B_u^yH_u(u_\rho) \ ,$$

hence

$$[B_u^y, \Psi]H_u(u_\rho) = \left(\dot{\Psi} - 2i\frac{yJ^y}{1+y\rho^2}\Psi\right)H_u(u_\rho).$$

It remains to prove that the left hand side of this equality is zero. We first show that, for any $p \in \mathbb{D}$ such that χ_p is a factor of χ, for every

$e \in E_u(\rho)$ such that $\chi_p e \in E_u(\rho)$, $[B_u^y, \chi_p](e) = 0$. We write

$$\begin{aligned} i[B_u^y, \chi_p](e) &= -yw^y\,(\Pi(\overline{w^y}\,\chi_p e) - \chi_p \Pi(\overline{w^y}\,e)) \\ &+ y^2 H_u w^y\,(\Pi(\overline{H_u w^y}\,\chi_p e) - \chi_p \Pi(\overline{H_u w^y}\,e)) \\ &- y^2((\chi_p e|H_u w^y) H_u w^y - \chi_p (e|H_u w^y) H_u w^y) \end{aligned}$$

We already used that, for any function $f \in L^2$, $\Pi(\chi_p f) - \chi_p \Pi(f)$ is proportional to $\frac{1}{1-\overline{p}z}$. Hence, we obtain

$$\begin{aligned} i[B_u^y, \chi_p](e) &= -yw^y \frac{c}{1-\overline{p}z} + y^2 H_u w^y \frac{\tilde{c}}{1-\overline{p}z} \\ &- y^2((\chi_p e|H_u w^y) H_u w^y - \chi_p (e|H_u w^y) H_u w^y) \end{aligned}$$

with

$$c = (\Pi(\overline{w^y}\,\chi_p e) - \chi_p \Pi(\overline{w^y}\,e)|1) = (\chi_p e|w^y) - (\chi_p|1)(e|w^y)$$

and

$$\tilde{c} = (\Pi(\overline{H_u w^y}\,\chi_p e) - \chi_p \Pi(\overline{H_u w^y}\,e)|1) = (\chi_p e|H_u(w^y)) - (\chi_p|1)(e|H_u(w^y))\ .$$

Now, for any $v \in E_u(\rho)$

$$(v|w^y) = (v|1\!\!1_{\{\rho^2\}}(H_u^2)(w^y)) = \frac{1}{1+y\rho^2}(v|1)$$

hence $c = 0$. On the other hand,

$$\begin{aligned} (v|H_u w^y) &= (v|1\!\!1_{\{\rho^2\}}(H_u^2)(H_u(w^y))) = \frac{1}{1+y\rho^2}(v|u_\rho) \\ &= \frac{1}{1+y\rho^2}(v|H_u(1)) = \frac{1}{1+y\rho^2}(1|H_u(v))\ . \end{aligned}$$

We infer

$$i[B_u^y, \chi_p](e) = C(z)\frac{1}{1+y\rho^2} y^2 H_u w^y$$

where

$$\begin{aligned} C(z) &= \frac{1}{1-\overline{p}z}((1|H_u(\chi_p e)) - (\chi_p|1)(1|H_u(e)) - (1|H_u(\chi_p e) + \chi_p(1|H_u(e)) \\ &= (1|H_u(\chi_p e))(\frac{1}{1-\overline{p}z} - 1) + (1|H_u(e))(\chi_p + \frac{p}{1-\overline{p}z}) \\ &= \frac{z}{1-\overline{p}z}(\overline{p}(1|H_u(\chi_p e)) + (1|H_u(e))) \ . \end{aligned}$$

We claim that $H_u(e) = \chi_p H_u(\chi_p e)$. Indeed, from the assumption $e \in E_u(\rho)$ and $\chi_p e \in E_u(\rho)$, we can write $e = f H_u(u_\rho)$ with $\Pi(\Psi\overline{f}) = \Psi\overline{f}$ and $\Pi(\Psi\overline{\chi_p f}) = \Psi\overline{\chi_p f}$. From Lemma 7, we infer

$$H_u(\chi_p e) = \rho\Psi\overline{\chi_p f} H_u(u_\rho) \ , \ H_u(e) = \rho\Psi\overline{f} H_u(u_\rho) \ .$$

This proves the claim. Since $(1|\chi_p) = -\overline{p}$, we conclude that $C(z) = 0$. Hence $[B_u^y, \chi_p](e) = 0$. Arguing as in the previous section, we conclude that $[B_u, \chi]H_u(u_\rho) = 0$.

It remains to consider the other eigenvalues. Let $\sigma \in \Sigma_K(u_0)$. Denote by u'_σ the orthogonal projection of u on $F_u(\sigma)$. We compute the derivative of $u'_\sigma = 1\!\!1_{\{\sigma^2\}}(K_u^2)(u)$ as before. From the Lax pair formula, we get

$$\begin{aligned} \frac{du'_\sigma}{dt} &= [C_u^y, 1\!\!1_{\{\sigma^2\}}(K_u^2)](u) + 1\!\!1_{\{\sigma^2\}}(K_u^2)[B_u^y, H_u](1) \\ &= C_u^y(u'_\sigma) + 1\!\!1_{\{\sigma^2\}}(K_u^2)(B_u^y(u) - C_u^y(u) - H_u(B_u^y(1))) \\ &= C_u^y(u'_\sigma) + 1\!\!1_{\{\sigma^2\}}(K_u^2)(iy^2(u|H_u w^y)H_u w^y + iyJ^y H_u w^y) \\ &= C_u^y(u'_\sigma) + iy1\!\!1_{\{\sigma^2\}}(K_u^2)H_u w^y \end{aligned}$$

since $(B_u^y - C_u^y)(h) = iy^2(h|H_u w^y)H_u w^y$ and $-yH_u^2 w^y = w^y - 1$ so that $(u| - yH_u w^y) = (-yH_u^2 w^y|1) = J^y - 1$.

We claim that

$$1\!\!1_{\{\sigma^2\}}(K_u^2)(H_u w^y) = \frac{J^y}{1+y\sigma^2}u'_\sigma \ . \tag{7.1.3}$$

Using $K_u^2 = H_u^2 - (\cdot|u)u$ one gets, for any $f \in L^2_+$

$$(I+yH_u^2)^{-1}f = (I+yK_u^2)^{-1}f - y((I+yH_u^2)^{-1}f|u)(I+yK_u^2)^{-1}u \ . \tag{7.1.4}$$

Applying formula (7.1.4) to $f = u$, we get

$$H_u w^y = (I+yH_u^2)^{-1}(u) = (I+yK_u^2)^{-1}(u) - y((I+yH_u^2)^{-1}(u)|u)(I+yK_u^2)^{-1}(u) \ ,$$

hence

$$H_u w^y = J^y (I + yK_u^2)^{-1}(u) \ . \tag{7.1.5}$$

Formula (7.1.3) follows by taking the orthogonal projection on $F_u(\sigma)$. Using Formula (7.1.3), we get

$$\frac{du'_\sigma}{dt} = C_u^y(u'_\sigma) + iy\frac{J^y}{1+y\sigma^2}u'_\sigma \ . \tag{7.1.6}$$

On the other hand, differentiating the equation

$$K_u(u'_\sigma) = \sigma\Psi u'_\sigma$$

one obtains

$$[C_u^y, K_u](u'_\sigma) + K_u\left(\frac{du'_\sigma}{dt}\right) = \sigma\dot{\Psi}u'_\sigma + \sigma\Psi\frac{du'_\sigma}{dt}.$$

From identity (7.1.5), we use the expression of $\frac{du'_\sigma}{dt}$ obtained above to get

$$\left(\dot{\Psi} + 2i\frac{yJ^y}{1+\sigma^2 y}\Psi\right)u'_\sigma = \sigma[C_u^y, \Psi](u'_\sigma) \ .$$

The result follows once we prove that $[C_u^y, \Psi](u'_\sigma) = 0$.

From the arguments developed before, it is sufficient to prove that $[C_u^y, \chi_p](f) = 0$ for any $f \in F_u(\sigma)$ such that $\chi_p f \in F_u(\sigma)$. As before

$$[C_u^y, \chi_p](f) = i\frac{c}{1-\overline{p}z}yw^y - iy^2 H_u w^y \frac{\tilde{c}}{1-\overline{p}z}$$

where

$$c = ((\chi_p - (\chi_p|1))f|w^y)$$

and

$$\tilde{c} = ((\chi_p - (\chi_p|1))f|H_u w^y).$$

Notice that $w^y = 1 - yH_u w^y$, hence $c = -y\tilde{c}$. Let us first prove that $\tilde{c} = 0$. Using formula (7.1.5),

$$\begin{aligned} \tilde{c} &= (\chi_p f|1\!\!1_{\{\sigma^2\}}(K_u^2)H_u w^y) - (\chi_p|1)(f|1\!\!1_{\{\sigma^2\}}(K_u^2)H_u w^y) \\ &= \frac{J^y}{1+y\sigma^2}((\chi_p - (\chi_p|1))f|u) = 0 \ , \end{aligned}$$

since, as we already observed at the end of the proof of Theorem 8,

$$F_u(\sigma) \cap 1^\perp = E_u(\sigma) = F_u(\sigma) \cap u^\perp \ .$$

This completes the proof. $\square$

We close this section by stating a corollary which will be useful for describing the symplectic form on $\mathcal{V}_{(d_1,\dots,d_n)}$.

Corollary 4. — *On $\mathcal{V}_{(d_1,\dots,d_n)}$, we have*

$$X_{J^y} = \sum_{r=1}^{n} (-1)^r \frac{2yJ^y}{1+ys_r^2} \frac{\partial}{\partial \psi_r} \ . \tag{7.1.7}$$

The vector fields $X_{J^y}, y \in \mathbb{R}_+$, generate an integrable sub-bundle of rank n of the tangent bundle of $\mathcal{V}_{(d_1,\dots,d_n)}$. The leaves of the corresponding foliation are the isotropic tori

$$\mathcal{T}((s_1,\dots,s_n),(\Psi_1,\dots,\Psi_n)) := \Phi^{-1}\left(\{(s_1,\dots,s_n)\} \times \mathbb{S}^1\Psi_1 \times \cdots \times \mathbb{S}^1\Psi_n\right) \ ,$$

where $(s_1,\dots,s_n) \in \Omega_n$ and $(\Psi_1,\dots,\Psi_n) \in \mathcal{B}^\sharp_{d_1} \times \cdots \times \mathcal{B}^\sharp_{d_n}$ are given.

Proof. — For every $y \in \mathbb{R}_+$, Theorem 11 can be rephrased as the following identities for Lie derivatives along X_{J^y}.

$$X_{J^y}(s_r) = 0 \ , \ X_{J^y}(\chi_r) = 0 \ , \ X_{J^y}(\psi_r) = (-1)^r \frac{2yJ^y}{1+ys_r^2} \ , \ r = 1,\dots,n \ .$$

This implies identity (7.1.7) on $\mathcal{V}_{(d_1,\dots,d_n)}$. Given n positive numbers $y_1 > \cdots > y_n$, the matrix

$$\left(\frac{1}{1+y_\ell s_r^2}\right)_{1\le \ell,r\le n}$$

is invertible. This implies that, for every $u \in \mathcal{V}_{(d_1,\dots,d_n)}$, the vector subspace of $T_u\mathcal{V}_{(d_1,\dots,d_n)}$ spanned by the $X_{J^y}(u), y \in \mathbb{R}_+$ is exactly

$$\operatorname{span}\left(\frac{\partial}{\partial \psi_r}, r = 1,\dots,n\right) \ .$$

The integrability follows, as well as the identification of the leaves, while the isotropy of the tori comes from the identity

$$\{J^y, J^{y'}\} = 0$$

which was proved in [11] and is also a consequence of identity 3.2.1 and of the conservation of the s_r's along the Hamiltonian curves of J^y, as stated in Theorem 11. □

7.2. The symplectic form on $\mathcal{V}_{(d_1,\dots,d_n)}$

In this section, we prove that the symplectic form ω restricted to $\mathcal{V}_{(d_1,\dots,d_n)}$ is given by

$$\omega = \sum_{r=1}^{n} d\left(\frac{s_r^2}{2}\right) \wedge d\psi_r \ . \tag{7.2.1}$$

Recall that the variable ψ_r is connected to the Blaschke product Ψ_r through the identity

$$\Psi_r = \mathrm{e}^{-i\psi_r}\chi_r \ ,$$

where χ_r is a Blaschke product built with a monic polynomial. Given an integer k, we denote by $\mathcal{B}_k^\sharp$ the submanifold of $\mathcal{B}_k$ made with Blaschke products built with monic polynomials of degree k.

Let us first point out that we get the following result as a corollary.

Corollary 5. — *The manifold $\mathcal{V}_{(d_1,\dots,d_n)}$ is an involutive submanifold of $\mathcal{V}(d)$, where*

$$d = 2\sum_{r=1}^{n} d_r + n.$$

Moreover, $\mathcal{V}_{(d_1,\dots,d_n)}$ is the disjoint union of the symplectic manifolds

$$\mathcal{W}(\chi_1,\dots,\chi_n) := \Phi^{-1}(\Omega_n \times (\mathbb{S}^1\chi_1 \times \cdots \times \mathbb{S}^1\chi_n)) \ ,$$

on which

$$\left(\frac{s_r^2}{2}, \psi_r\right)_{1\le r\le n}$$

are action angle variables for the cubic Szegő flow.

Proof. — From the definition of an involutive submanifold, one has to prove that, at every point u of $\mathcal{V}_{(d_1,\dots,d_n)}$, the tangent space $T_u\mathcal{V}_{(d_1,\dots,d_n)}$ contains its orthogonal relatively to ω. We use an argument of dimension. Namely, one has

$$\begin{aligned}\dim_{\mathbb{R}}(T_u\mathcal{V}_{(d_1,\dots,d_n)})^\perp &= \dim_{\mathbb{R}} T_u\mathcal{V}(d) - \dim_{\mathbb{R}} T_u\mathcal{V}_{(d_1,\dots,d_n)} \\ &= 2d - (2n + 2\sum_{r=1}^{n} d_r) = 2\sum_{r=1}^{n} d_r \ .\end{aligned}$$

One the other hand, from equation (7.2.1), the tangent space to the manifold

$$\mathcal{F}(u) := \Phi^{-1}\left(\{(s_r(u))\} \times \prod_{r=1}^{n} \mathrm{e}^{-i\psi_r(u)}\mathcal{B}^{\sharp}_{d_r}\right)$$

is clearly a subset of $(T_u\mathcal{V}_{(d_1,\ldots,d_n)})^{\perp}$. Since its dimension equals $2\sum d_r$, we get the equality and hence the first result. The second result is an immediate consequence of the previous sections. □

Remark 4. — *As this is the case for any involutive submanifold of a symplectic manifold, the subbundle $(T\mathcal{V}_{(d_1,\ldots,d_n)})^{\perp}$ of $T\mathcal{V}_{(d_1,\ldots,d_n)}$ is integrable. The leaves of the corresponding isotropic foliation are the manifolds $\mathcal{F}(u)$ above.*

Now, we prove equality (7.2.1). We first establish the following lemma, as a consequence of Theorem 11.

Lemma 19. — *On $\mathcal{V}_{(d_1,\ldots,d_n)}$,*

$$\omega = \sum_{r=1}^{n} d\left(\frac{s_r^2}{2}\right) \wedge d\psi_r + \tilde{\omega} \ .$$

where, for any $1 \le r \le n$,

$$i_{\frac{\partial}{\partial\psi_r}}\tilde{\omega} = 0 \ .$$

Proof. — Taking the interior product of both sides of identity (7.1.7) with the restriction of ω to $\mathcal{V}_{(d_1,\ldots,d_n)}$, we obtain

$$-d(\log J^y) = \sum_{r=1}^{n}(-1)^r \frac{2y}{1+ys_r^2} i_{\frac{\partial}{\partial\psi_r}}\omega \ .$$

On the other hand, from formula (3.2.7),

$$d(\log(J^y)) = \sum_{r=1}^{n}(-1)^r \frac{2y}{1+ys_r^2} d\left(\frac{s_r^2}{2}\right) \ .$$

Identification of residues in the y variables yields

$$d\left(\frac{s_r^2}{2}\right) = -i_{\frac{\partial}{\partial\psi_r}}\omega \ , \ r = 1, \ldots, n \ .$$

Since

$$i_{\frac{\partial}{\partial\psi_r}}\left(\sum_{r'=1}^{n} d\left(\frac{s_{r'}^2}{2}\right)\wedge d\psi_{r'}\right) = -d\left(\frac{s_r^2}{2}\right)\ ,$$

this completes the proof. $\square$

Since $d\omega = 0$, we have $d\tilde{\omega} = 0$. Combining this information with $i_{\frac{\partial}{\partial\psi_r}}\tilde{\omega} = 0$, we conclude that

$$\tilde{\omega} = \pi^*\beta\ ,$$

where β is a closed 2-form on $\Omega_n \times \prod_{r=1}^{n} \mathcal{B}_{d_r}^{\sharp}$, and

$$\pi(u) := ((s_r(u))_{1\le r\le n}, (\chi_r(u)_{1\le r\le n})\ .$$

In order to prove that $\tilde{\omega} = 0$, it is therefore sufficient to prove that $\tilde{\omega} = 0$ on the submanifold

$$\mathcal{V}_{(d_1,\dots,d_n),\mathrm{red}} := \Phi^{-1}\left(\Omega_n \times \prod_{r=1}^{n} \mathcal{B}_{d_r}^{\sharp}\right)$$

given by the equations $\psi_r = 0$, $r = 1,\dots,n$.

Lemma 20. — *The restriction of ω to $\mathcal{V}_{(d_1,\dots,d_n),\mathrm{red}}$ is 0.*

Proof. — Consider the differential form α of of degree 1 defined

$$\langle\alpha(u), h\rangle := \mathrm{Im}(u|h)\ .$$

It is elementary to check that

$$\frac{1}{2}d\alpha = \omega\ ,$$

hence the statement is consequence of the fact that the restriction of α to $\mathcal{V}_{(d_1,\dots,d_n),\mathrm{red}}$ is 0. Let us prove this stronger fact. By a density argument, we may assume that $n = 2q$ is even, and that the Blaschke products $\chi_r(u)$ have only simple zeroes. Firstly, we describe the tangent space of $\mathcal{V}_{(d_1,\dots,d_n),\mathrm{red}}$ at a generic point. We use the notation of section 4.1.2.

Lemma 21. — *The tangent vectors to $\mathcal{V}_{(d_1,\dots,d_n),\mathrm{red}}$ at a generic point u where every χ_r has only simple zeroes are linear combinations with real*

coefficients of $u_j, u_jH_u(u_\ell), 1 \le j, \ell \le q$, *and of the following functions, for* $\zeta \in \mathbb{C}$ *and* $1 \le j, k \le q$,

$$\begin{aligned} \dot{u}_{\chi_{2j-1},\zeta}(z) &:= \left(\overline{\zeta}\frac{z}{1-\overline{p}z} - \zeta\frac{1}{z-p}\right) u_j(z)H_u(u_j)(z)\ ,\ \chi_{2j-1}(p) = 0\ , \\ \dot{u}_{\chi_{2k},\zeta}(z) &:= \left(\overline{\zeta}\frac{z}{1-\overline{p}z} - \zeta\frac{1}{z-p}\right) zu_k'(z)K_u(u_k')(z)\ ,\ \chi_{2k}(p) = 0\ . \end{aligned}$$

We assume Lemma 21 and show how it implies Lemma 20. Notice that

$$\begin{aligned} (u|u_j) &= \|u_j\|^2\ ,\ (u|u_jH_u(u_j)) = \|H_u(u_j)\|^2\ , \\ (u|H_u(u_\ell)u_j) &= (H_u(u_j)|H_u(u_\ell)) = 0\ ,\ j \ne \ell\ , \end{aligned}$$

and therefore $\alpha(u)$ cancels on $u_j, u_jH_u(u_\ell), 1 \le j, \ell \le q$. We now deal with vectors $\dot{u}_{\chi_r,\zeta}$.

$$\begin{aligned} (u|\dot{u}_{\chi_{2j-1},\zeta}) &= \zeta\left(u\middle|\frac{z}{1-\overline{p}z}u_jH_u(u_j)\right) - \overline{\zeta}\left(u\middle|\frac{u_j}{z-p}H_u(u_j)\right) \\ &= \zeta\left(H_u(u_j)\middle|\frac{z}{1-\overline{p}z}H_u(u_j)\right) - \overline{\zeta}\left(H_u^2(u_j)\middle|\frac{u_j}{z-p}\right)\ , \end{aligned}$$

where we used that $u_j/(z-p)$ belongs to L^2_+. Since $H_u^2(u_j) = \rho_j^2 u_j$ and $\rho_j^2|u_j|^2 = |H_u(u_j)|^2$ on the unit circle, we infer

$$(u|\dot{u}_{\chi_{2j-1},\zeta}) = \zeta\left(H_u(u_j)\middle|\frac{z}{1-\overline{p}z}H_u(u_j)\right) - \overline{\zeta}\left(\frac{z}{1-\overline{p}z}H_u(u_j)\middle|H_u(u_j)\right)$$

and consequently

$$\langle\alpha(u), \dot{u}_{\chi_{2j-1},\zeta}\rangle = 2\mathrm{Im}\ \zeta\left(H_u(u_j)\middle|\frac{z}{1-\overline{p}z}H_u(u_j)\right)\ ,$$

which is 0 for every $\zeta \in \mathbb{C}$ if and only if

$$\left(H_u(u_j)\middle|\frac{z}{1-\overline{p}z}H_u(u_j)\right) = 0\ .$$

Let us prove this identity. Set

$$v := \frac{z}{1-\overline{p}z}H_u(u_j)\ .$$

Notice that, since $\chi_{2j-1}(p) = 0$, $v \in E_u(\rho_j)$, and moreover

$$(v|1) = v(0) = 0\ .$$

Therefore

$$(H_u(u_j)|v) = (H_u(v)|u_j) = (H_u(v)|u) = (1|H_u^2(v)) = \rho_j^2(1|v) = 0 \ .$$

We conclude that

$$\langle \alpha(u), \dot{u}_{\chi_{2j-1},\zeta}\rangle = 0 \ .$$

Similarly, we calculate

$$\begin{aligned}(u|\dot{u}_{\chi_{2k},\zeta}) &= \zeta\left(u\Big|\frac{z}{1-\overline{p}z}zu_k'K_u(u_k')\right) - \overline{\zeta}\left(u\Big|\frac{K_u(u_k')}{z-p}zu_k'\right)\\ &= \zeta\left(K_u(u_k')\Big|\frac{z}{1-\overline{p}z}K_u(u_k')\right) - \overline{\zeta}\left(K_u(u_k')\Big|\frac{K_u(u_k')}{z-p}\right) \ ,\end{aligned}$$

where we have used that $K_u(u_k')/(z-p)$ belongs to L^2_+. We conclude that

$$\langle \alpha(u), \dot{u}_{\chi_{2k},\zeta}\rangle = 2\mathrm{Im}\ \zeta\left(K_u(u_k')\Big|\frac{z}{1-\overline{p}z}K_u(u_k')\right) \ ,$$

which is 0 for every $\zeta\in\mathbb{C}$ if and only if

$$\left(K_u(u_k')\Big|\frac{z}{1-\overline{p}z}K_u(u_k')\right) = 0 \ .$$

Since $|K_u(u_k')|^2 = \sigma_k^2|u_k'|^2$ on the unit circle, we are left to prove

$$\left(u_k'\Big|\frac{z}{1-\overline{p}z}u_k'\right) = 0 \ .$$

Set

$$w := \frac{1}{1-\overline{p}z}u_k' \ .$$

We notice that $w\in F_u(\sigma_k)$, and that $zw\in F_u(\sigma_k)$. Moreover,

$$w = \frac{1}{1-|p|^2}(\overline{p}\chi_p + 1)u_k' \ ,$$

therefore, setting $\chi_{2k} := g_k\chi_p$,

$$K_u(w) = \frac{\sigma_k}{1-|p|^2}(pg_ku_k' + \chi_{2k}u_k') = \frac{\sigma_k g_k}{1-|p|^2}(p+\chi_p)u_k' = \sigma_k g_k zw \ .$$

In particular,

$$(K_u(w)|1) = K_u(w)(0) = 0 \ .$$

We now conclude as follows,

$$\left(u_k'\Big|\frac{z}{1-\overline{p}z}u_k'\right) = (u_k'|zw) = (u|zw) = (K_u(w)|1) = 0 \ .$$

This completes the proof up to the proof of lemma 21. □

Let us prove lemma 21. We are going to use formulae (1.0.5), namely

$$u(z) = \langle \mathscr{C}(z)^{-1}(\chi_{2j-1}(z))_{1\le j\le q}, \mathbf{1}\rangle,$$

where

$$\mathscr{C}(z) = \left(\frac{\rho_j - \sigma_k z\chi_{2k}(z)\chi_{2j-1}(z)}{\rho_j^2 - \sigma_k^2}\right)_{1\le j,k\le q} .$$

If we denote by $\dot{}$ the derivative with respect to one of the parameters ρ_j, σ_k or one of the coefficients of the χ_r, we have

$$\dot u(z) = -\langle \mathscr{C}(z)^{-1}\dot{\mathscr{C}}(z)\mathscr{C}(z)^{-1}(\chi_{2j-1}(z))_{1\le j\le q}, \mathbf{1}\rangle + \langle \mathscr{C}(z)^{-1}(\dot\chi_{2j-1}(z))_{1\le j\le q}, \mathbf{1}\rangle .$$

In the case of the derivative with respect to ρ_j,

$$\dot{\mathscr{C}}(z)_{ik} = -\frac{\rho_j^2 + \sigma_k^2 - 2\sigma_k\rho_j z\chi_{2j-1}(z)\chi_{2k}(z)}{(\rho_j^2 - \sigma_k^2)^2}\delta_{ij} ,$$

and, using

$$\mathscr{C}(z)^{-1}(\chi_{2j-1}(z))_{1\le j\le q} = (u'_k(z))_{1\le k\le q} , \quad {}^t\mathscr{C}(z)^{-1}(\mathbf{1}) = (h_j(z))_{1\le j\le q} ,$$

one gets

$$\begin{aligned}\dot u(z) &= h_j(z)\sum_{k=1}^q u'_k(z)\frac{(\rho_j^2 + \sigma_k^2 - 2\sigma_k\rho_j z\chi_{2j-1}(z)\chi_{2k}(z))}{(\rho_j^2 - \sigma_k^2)^2}\\ &= \frac{H_u(u_j)(z)}{\rho_j}\sum_{k=1}^q \frac{\rho_j^2 + \sigma_k^2}{(\rho_j^2 - \sigma_k^2)^2}u'_k(z) - 2\rho_j u_j(z)\sum_{k=1}^q \frac{zK_u(u'_k)(z)}{(\rho_j^2 - \sigma_k^2)^2} .\end{aligned}$$

Observe that

$$zK_u(u'_k)(z) = [SS^*H_u(u'_k)](z) = H_u(u'_k)(z) - \kappa_k^2 ,$$

and that u'_k is a linear combination with real coefficients of u_ℓ in view of (3.1.6). We infer that, in this case, $\dot u$ is a linear combination with real coefficients of u_j and $u_m H_u(u_\ell)$.

In the case of the derivative with respect to σ_k, one similarly gets

$$\begin{aligned}\dot u(z) &= u'_k(z)\sum_{j=1}^q h_j(z)\frac{z\chi_{2j-1}(z)\chi_{2k}(z)(\rho_j^2 + \sigma_k^2) - 2\sigma_k\rho_j}{(\rho_j^2 - \sigma_k^2)^2}\\ &= \frac{zK_u(u'_k)(z)}{\sigma_k}\sum_{j=1}^q \frac{(\rho_j^2 + \sigma_k^2)u_j(z)}{(\rho_j^2 - \sigma_k^2)^2} - 2\sigma_k u'_k(z)\sum_{j=1}^q \frac{H_u(u_j)(z)}{(\rho_j^2 - \sigma_k^2)^2} ,\end{aligned}$$

which is a linear combination with real coefficients of u_j and $u_j H_u(u_\ell)$.

In the case of a derivative with respect to one of the zeroes of χ_{2j-1}, we obtain

$$\begin{aligned}
\dot{u}(z) &= \sum_{k=1}^{q} u_k'(z) h_j(z) \frac{\sigma_k z \dot{\chi}_{2j-1}(z)\chi_{2k}(z)}{\rho_j^2-\sigma_k^2} + h_j(z)\dot{\chi}_{2j-1}(z) \\
&= \frac{\dot{\chi}_{2j-1}(z)}{\chi_{2j-1}(z)} u_j(z) \left[\sum_{k=1}^{q} u_k'(z)\frac{\sigma_k z \chi_{2k}(z)}{\rho_j^2-\sigma_k^2} + 1\right] \\
&= \frac{\dot{\chi}_{2j-1}(z)}{\chi_{2j-1}(z)} u_j(z) \left(z\frac{K_u(u_j)}{\tau_j^2} + 1\right) \\
&= \frac{\dot{\chi}_{2j-1}(z)}{\chi_{2j-1}(z)} u_j(z) \frac{H_u(u_j)}{\tau_j^2} \ .
\end{aligned}$$

In the case of a derivative with respect to one of the zeroes of χ_{2k}, we obtain similarly

$$\dot{u}(z) = \frac{\dot{\chi}_{2k}(z)}{\chi_{2k}(z)} \frac{u_k'(z) z K_u(u_k')(z)}{\kappa_k^2} \ .$$

The proof of Lemma 21 is completed by observing that, since χ_r is a product of functions χ_p for $|p|<1$, $\dot{\chi}_r/\chi_r$ is a sum of terms of the form

$$\left(\overline{\zeta}\frac{z}{1-\overline{p}z} - \zeta\frac{1}{z-p}\right)$$

where $\zeta := \dot{p}$.

7.3. Invariant tori of the Szegő hierarchy and unitary equivalence of pairs of Hankel operators

In this section, we identify the sets of symbols $u \in VMO_+ \setminus \{0\}$ having the same list of singular values (s_r) and the same list (χ_r) of monic Blaschke products, for the pair (H_u, K_u). In view of Theorems 5 and 6, these sets are tori. In the finite dimensional case, they are precisely the Lagrangian tori of the symplectic manifolds $W(\chi_1, \dots, \chi_n)$ of Corollary 5, obtained by freezing the action variables. Moreover, $VMO_+ \setminus \{0\}$ is the disjoint union of these tori, and, from sections 5.1 and 7.1, the Hamilton flows of the Szegő hierarchy act on them. We prove that these tori are

classes of some specific unitary equivalence between the pairs (H_u, K_u), which we now define.

Definition 2. — *Given* $u, \tilde{u} \in VMO_+ \setminus \{0\}$, *we set* $u \sim \tilde{u}$ *if there exist unitary operators* U, V *on* L^2_+ *such that*

$$H_{\tilde{u}} = UH_uU^* \ , \ K_{\tilde{u}} = VK_uV^* \ ,$$

and there exists a Borel function $F : \mathbb{R}_+ \to \mathbb{S}^1$ *such that*

$$U(u) = F(H^2_{\tilde{u}})\tilde{u} \ , \ V(u) = F(K^2_{\tilde{u}})\tilde{u} \ , \ U^*V = \overline{F}(H^2_u)F(K^2_u) \ .$$

It is easy to check that the above definition gives rise to an equivalence relation.

Theorem 12. — *Given* $u, \tilde{u} \in VMO_+ \setminus \{0\}$, *the following assertions are equivalent.*

1. $u \sim \tilde{u}$.
2. $\forall r \geq 1, s_r(u) = s_r(\tilde{u})$ *and* $\exists \gamma_r \in \mathbb{T} : \Psi_r(\tilde{u}) = \mathrm{e}^{i\gamma_r}\Psi_r(u)$.

Proof. — Assume that (1) holds. Then $H^2_{\tilde{u}}$ is unitarily equivalent to H^2_u, and $K^2_{\tilde{u}}$ is unitarily equivalent to K^2_u. This clearly implies $\Sigma_H(\tilde{u}) = \Sigma_H(u)$ and $\Sigma_K(\tilde{u}) = \Sigma_K(u)$, so that $s_r(\tilde{u}) = s_r(u)$ for every r. Let us show that, for every r, $\Psi_r(u)$ and $\Psi_r(\tilde{u})$ only differ by a phase factor. Of course the only cases to be addressed are $d_r \geq 1$. We start with $r = 2j - 1$. From the hypothesis, we infer

$$U(u_j) = U(\mathbf{1}_{\{\rho_j^2\}}(H^2_u)(u)) = \mathbf{1}_{\{\rho_j^2\}}(H^2_{\tilde{u}})(U(u)) = F(\rho_j^2)\tilde{u}_j \ ,$$

and, consequently,

$$U(H_u(u_j)) = H_{\tilde{u}}(U(u_j)) = \overline{F}(\rho_j^2)H_{\tilde{u}}(\tilde{u}_j) \ . \tag{7.3.1}$$

Next we take advantage of the identity

$$U^*V = \overline{F}(H^2_u)F(K^2_u) \ ,$$

by evaluating U^*S^*U on the closed range of H_u. We compute

$$\begin{aligned} U^*S^*UH_u &= U^*S^*H_{\tilde{u}}U = U^*K_{\tilde{u}}U = U^*VK_uV^*U \\ &= \overline{F}(H^2_u)F(K^2_u)K_u\overline{F}(K^2_u)F(H^2_u) = \overline{F}(H^2_u)F(K^2_u)^2K_uF(H^2_u) \\ &= \overline{F}(H^2_u)F(K^2_u)^2S^*\overline{F}(H^2_u)H_u \ , \end{aligned}$$

and we conclude, on $\overline{\mathrm{Ran}(H_u)}$,

$$U^*S^*U = \overline{F}(H_u^2)F(K_u^2)^2S^*\overline{F}(H_u^2)\ . \tag{7.3.2}$$

For simplicity, set $D := D_{2j-1}$ and $d := d_{2j-1}$. Recall from Proposition 3 that a basis of $E_u(\rho_j)$ is

$$\left(\frac{z^a}{D}H_u(u_j)\ ,\ a = 0, \dots, d\right),$$

and a basis of $F_u(\rho_j) = E_u(\rho_j) \cap u^\perp$ is

$$\left(\frac{z^b}{D}H_u(u_j)\ ,\ b = 0, \dots, d-1\right).$$

For $a = 1, \dots, d_{2j-1}$, we infer

$$U^*S^*U\left(\frac{z^a}{D}H_u(u_j)\right) = \frac{z^{a-1}}{D}H_u(u_j)\ ,$$

or

$$U\left(\frac{z^a}{D}H_u(u_j)\right) = (S^*)^{d-a}U\left(\frac{z^d}{D}H_u(u_j)\right)\ ,\ a = 0, \dots, d\ .$$

This implies, for $a = 0, \dots, d-1$, that the right hand side belongs to $F_{\tilde{u}}(\rho_j)$. On the other hand, if $P \in \mathbb{C}[z]$ has degree at most d, one easily checks that

$$S^*\left(\frac{P}{\tilde{D}}H_{\tilde{u}}(\tilde{u}_j)\right) = P(0)K_{\tilde{u}}(\tilde{u}_j) + R\ ,\ R \in F_{\tilde{u}}(\rho_j)\ .$$

Notice that the right hand side belongs to $F_{\tilde{u}}(\rho_j)$ if and only if $K_{\tilde{u}}(\tilde{u}_j) \in F_{\tilde{u}}(\rho_j)$ or $P(0) = 0$. Assume for a while that $K_{\tilde{u}}(\tilde{u}_j)$ does not belong to $F_{\tilde{u}}(\rho_j)$. Then, writing

$$U\left(\frac{z^d}{D}H_u(u_j)\right) = \frac{P}{\tilde{D}}H_{\tilde{u}}(\tilde{u}_j)\ ,$$

and using that, for $a = 0, \dots, d-1$,

$$(S^*)^{d-a}\left(\frac{P}{\tilde{D}}H_{\tilde{u}}(\tilde{u}_j)\right) \in F_{\tilde{u}}(\rho_j)\ ,$$

we infer $P(0) = 0$, and, by iterating this argument, that P is divisible by z^d, in other words,

$$U\left(\frac{z^d}{D}H_u(u_j)\right) = c\frac{z^d}{\tilde{D}}H_{\tilde{u}}(\tilde{u}_j)\ ,$$

for some $c \in \mathbb{C}$, and conclude

$$U\left(\frac{z^a}{D}H_u(u_j)\right) = c\frac{z^a}{\tilde{D}}H_{\tilde{u}}(\tilde{u}_j)\ ,\ a = 0, \dots, d\ .$$

Comparing to formula (7.3.1) for $U(H_u(u_j))$, we obtain

$$cD(z) = \overline{F}(\rho_j^2)\tilde{D}(z)\ .$$

Since $D(0) = 1 = \tilde{D}(0)$, we conclude $c = \overline{F}(\rho_j^2)$, $D = \tilde{D}$, and finally

$$\Psi_{2j-1}(\tilde{u}) = \overline{F}(\rho_j^2)^2\Psi_{2j-1}(u)\ .$$

We now turn to study the special case $K_{\tilde{u}}(\tilde{u}_j) \in F_{\tilde{u}}(\rho_j)$. This reads

$$0 = (K_{\tilde{u}}^2 - \rho_j^2 I)K_{\tilde{u}}(\tilde{u}_j) = K_{\tilde{u}}((H_{\tilde{u}}^2 - \rho_j^2 I)\tilde{u}_j - \|\tilde{u}_j\|^2\tilde{u}) = -\|\tilde{u}_j\|^2 K_{\tilde{u}}(\tilde{u})\ .$$

In other words, this imposes $K_{\tilde{u}}(\tilde{u}) = 0$, or $\tilde{u} = \rho\tilde{\Psi}$, where $\tilde{\Psi}$ is a Blaschke product of degree d. Making V^* act on the identity $K_{\tilde{u}}(\tilde{u}) = 0$, we similarly conclude $u = \rho\Psi$, where Ψ is a Blaschke product of degree d, so what we have to check is that Ψ and $\tilde{\Psi}$ only differ by a phase factor. In this case, S^* sends $E_u(\rho) = \mathrm{Ran}(H_u)$ into $F_u(\rho)$, so that (7.3.2) becomes, on $\mathrm{Ran}(H_u)$,

$$U^*S^*U = S^*\ .$$

In other words, the actions of S^* on $W := \mathrm{span}\left(\frac{z^a}{D}, a = 0, \dots, d\right)$ and on $\tilde{W} := \mathrm{span}\left(\frac{z^a}{\tilde{D}}, a = 0, \dots, d\right)$ are conjugated. Writing

$$D(z) = \prod_{p\in\mathcal{P}}(1 - \overline{p}z)^{m_p}\ ,$$

where $\mathcal{P}$ is a finite subset of $\mathbb{D}\setminus\{0\}$, and m_p are positive integers, and using the elementary identities

$$(S^* - \overline{p}I)\left(\frac{1}{(1-\overline{p}z)^k}\right) = S^*\left(\frac{1}{(1-\overline{p}z)^{k-1}}\right)\ ,$$

one easily checks that the eigenvalues of S^* on W are precisely the $\overline{p}$'s, for $p \in \mathcal{P}$, and 0, with the corresponding algebraic multiplicities m_p and

$$m_0 = 1 + d - \sum_{p\in\mathcal{P}} m_p\ .$$

We conclude that $D = \tilde{D}$, whence the claim.

Next, we study the case $r = 2k$. Then

$$V(u'_k) = F(\sigma_k^2)\tilde{u}'_k \ , \ V(K_u(u'_k)) = \overline{F}(\sigma_k^2)K_{\tilde{u}}(\tilde{u}'_k) \ .$$

Denote by P_u the orthogonal projector onto $\overline{\mathrm{Ran}(H_u)}$, and compute

$$\begin{aligned} V^*P_{\tilde{u}}SVK_u &= V^*P_{\tilde{u}}SK_{\tilde{u}}V = V^*(H_{\tilde{u}} - (\tilde{u}|\,.\,)P_{\tilde{u}}(1))V \\ &= V^*U(H_u - (U^*(\tilde{u})|\,.\,)U^*(P_{\tilde{u}}(1)))U^*V \\ &= \overline{F}(K_u^2)F(H_u^2)(H_u - (\overline{F}(H_u^2)u|\,.\,)F(H_u^2)P_u(1))\overline{F}(H_u^2)F(K_u^2) \\ &= \overline{F}(K_u^2)F(H_u^2)^2(H_u - (u|\,.\,)P_u(1))F(K_u^2) \\ &= \overline{F}(K_u^2)F(H_u^2)^2P_uSK_uF(K_u^2) = \overline{F}(K_u^2)F(H_u^2)^2P_uS\overline{F}(K_u^2)K_u \ , \end{aligned}$$

so that, on $\overline{\mathrm{Ran}(K_u)}$,

$$V^*P_{\tilde{u}}SV = \overline{F}(K_u^2)F(H_u^2)^2P_uS\overline{F}(K_u^2) \ . \tag{7.3.3}$$

For simplicity again, set $D := D_{2k}$ and $d := d_{2k}$. Recall from proposition 3 that a basis of $F_u(\sigma_k)$ is

$$\left(\frac{z^a}{D}u'_k \ , \ a = 0, \dots, d\right) ,$$

and a basis of $E_u(\sigma_k) = F_u(\sigma_k) \cap u^\perp$ is

$$\left(\frac{z^a}{D}u'_k \ , \ a = 1, \dots, d\right) \ .$$

Applying identity (7.3.3) to $\frac{z^a}{D}u'_k$ for $a = 0, \dots, d-1$, we infer

$$V\left(\frac{z^a}{D}u'_k\right) = (P_{\tilde{u}}S)^aV\left(\frac{1}{D}u'_k\right) \ , \ a = 0, \dots, d \ .$$

In particular, the right hand side belongs to $E_{\tilde{u}}(\sigma_k)$ for $a = 1, \dots, d$. On the other hand, if $Q \in \mathbb{C}[z]$ has degree at most d,

$$P_{\tilde{u}}S\left(\frac{Q}{\tilde{D}}\tilde{u}'_k\right) = \gamma P_{\tilde{u}}SK_{\tilde{u}}(\tilde{u}'_k) + R \ , \ R \in E_{\tilde{u}}(\sigma_k) \ ,$$

where $\gamma\sigma_k\mathrm{e}^{-i\tilde{\psi}_{2k}}$ is the coefficient of z^d in Q. Therefore the left hand side belongs to $E_{\tilde{u}}(\sigma_k)$ if and only if

$$\begin{aligned} 0 &= \gamma(H_{\tilde{u}}^2 - \sigma_k^2I)P_{\tilde{u}}SK_{\tilde{u}}(\tilde{u}'_k) \\ &= \gamma(H_{\tilde{u}}^2 - \sigma_k^2I)(H_{\tilde{u}}(\tilde{u}'_k) - \|\tilde{u}'_k\|^2P_{\tilde{u}}(1)) \\ &= \gamma\sigma_k^2\|\tilde{u}'_k\|^2P_{\tilde{u}}(1) \ , \end{aligned}$$

which is impossible. We conclude that $\gamma = 0$, which means that the degree of Q is at most $d-1$. Iterating this argument, we infer

$$V\left(\frac{1}{D}u'_k\right) = c\frac{1}{\tilde{D}}\tilde{u}'_k \ ,$$

for some $c \in \mathbb{C}$, and finally

$$V\left(\frac{z^a}{D}u'_k\right) = c\frac{z^a}{\tilde{D}}\tilde{u}'_k \ , \ a = 0, \dots d \ .$$

Comparing to the above formula for $V(u'_k)$, we obtain

$$cD = F(\sigma_k^2)\tilde{D} \ ,$$

thus, since $D(0) = 1 = \tilde{D}(0)$, we have $D = \tilde{D}$, $c = F(\sigma_k^2)$, and finally $\Psi_{2k}(\tilde{u}) = F(\sigma_k^2)^2\Psi_{2k}(u)$.

Assume that (2) holds. Define $F : \Sigma_H(u) \cup \Sigma_K(u) \to \mathbb{S}^1$ by

$$F(\rho_j^2) = \mathrm{e}^{-i\frac{\gamma_{2j-1}}{2}} \ ; \ F(\sigma_k^2) = \mathrm{e}^{i\frac{\gamma_{2k}}{2}} \ ,$$

and, if necessary, we define $F(0)$ to be any complex number of modulus 1. Next we define U on the closed range of H_u, which is the closed orthogonal sum of $E_u(s_r)$. Thus we just have to define

$$U : E_u(s_r) \to E_{\tilde{u}}(s_r).$$

If $r = 2j-1$, we set

$$U\left(\frac{z^a}{D_{2j-1}}H_u(u_j)\right) = \overline{F}(\rho_j^2)\frac{z^a}{D_{2j-1}}H_{\tilde{u}}(\tilde{u}_j) \ , \ a = 0, \dots, d_{2j-1} \ . \tag{7.3.4}$$

If $r = 2k$ and $d_{2k} \geq 1$, we set

$$U\left(\frac{z^b}{D_{2k}}u'_k\right) = F(\sigma_k^2)\frac{z^b}{D_{2k}}\tilde{u}'_k \ , \ b = 1, \dots, d_{2k} \ . \tag{7.3.5}$$

Using (7.3.4) we obtain

$$\begin{aligned} U(u_j) &= \frac{1}{\rho_j}U(\Psi_{2j-1}(u)H_u(u_j)) = \frac{1}{\rho_j}\overline{F}(\rho_j^2)\Psi_{2j-1}(u)H_{\tilde{u}}(\tilde{u}_j) \\ &= \frac{1}{\rho_j}F(\rho_j^2)\Psi_{2j-1}(\tilde{u})H_{\tilde{u}}(\tilde{u}_j) = F(\rho_j^2)\tilde{u}_j \ . \end{aligned}$$

Consequently, we get

$$U(u)=\sum_j U(u_j)=\sum_j F(\rho_j^2)\tilde u_j = F(H_{\tilde u}^2)\tilde u\ .$$

A similar argument combined to Proposition 3 leads to

$$UH_u=H_{\tilde u}U.$$

Next, we prove that U is unitary. It is enough to prove that every map $U:E_u(s_r)\to E_{\tilde u}(s_r)$ is unitary, or that the Gram matrix of a basis of $E_u(s_r)$ is equal to the Gram matrix of its image. We first deal with $r=2j-1$. Equivalently, we prove that, for $a,b=0,\dots,d_{2j-1}-1$,

$$\left(\frac{z^a}{D_{2j-1}}H_u(u_j)|\frac{z^b}{D_{2j-1}}H_u(u_j)\right)=\left(\frac{z^a}{D_{2j-1}}H_{\tilde u}(\tilde u_j)|\frac{z^b}{D_{2j-1}}H_{\tilde u}(\tilde u_j)\right)\ .$$

We set

$$\zeta_{a-b}:=\left(\frac{z^a}{D_{2j-1}}H_u(u_j)|\frac{z^b}{D_{2j-1}}H_u(u_j)\right),\ a,b=0,\dots,d_{2j-1}-1\ ,$$

and we notice that $\zeta_{-k}=\overline{\zeta}_k\ ,\ k=-d_{2j-1},\dots,d_{2j-1}$. We drop the subscript $2j-1$ for simplicity and we set

$$D(z):=1+\overline{a}_1z+\dots+\overline{a}_dz^d.$$

As $\Psi H_u(u_j)$ is orthogonal to $\frac{z^a}{D}H_u(u_j)$ for $a=0,\dots,d-1$, and $\|H_u(u_j)\|^2=\rho_j^2\tau_j^2$, we obtain the system

$$\left\{\begin{array}{l}\zeta_{d-b}+a_1\zeta_{d-b-1}+\dots+a_d\zeta_{-b}=0\ ,\ b=0,\dots,d-1\ ,\\ \zeta_0+a_1\zeta_{-1}+\dots+a_d\zeta_{-d}=\rho_j^2\tau_j^2\ .\end{array}\right.\tag{7.3.6}$$

Lemma 22. — *Let $a_1,\dots a_d$ be complex numbers such that the polynomial $z^d+a_1z^{d-1}+\dots+a_d$ has all its roots in $\mathbb{D}$. Then the system (7.3.6) has at most one solution ζ_k, $k=-d\dots,d$ with $\overline{\zeta}_k=\zeta_{-k}$.*

Assume for a while that this lemma is proved. Since τ_j^2 can be expressed in terms of the (s_r)'s — see (3.2.5), we infer that $U:E_u(\rho_j)\to E_{\tilde u}(\rho_j)$ is unitary. Similarly, one proves that the Gram matrix of the basis

$$\frac{z^a}{D_{2k}}u_k',\ a=0,\dots,d_{2k}$$

of $F_u(\sigma_k)$ only depends on the (s_r)'s and on D_{2k}. In particular,

$$U:E_u(\sigma_k)\to E_{\tilde u}(\sigma_k)$$

is unitary and finally is unitary from the closed range of H_u onto the closed range of $H_{\tilde{u}}$.

Next, we construct V on the closed range of H_u which is the orthogonal sum of the $F_u(\sigma)$ for $\sigma \in \Sigma_H \cup \Sigma_K$. Thus we just have to define $V : F_u(\sigma) \to E_{\tilde{u}}(\sigma)$ for $\sigma \in \Sigma_H \cup \Sigma_K$.

If $r = 2j - 1$, we set

$$V\left(\frac{z^a}{D_{2j-1}} H_u(u_j)\right) = \overline{F}(\rho_j^2) \frac{z^a}{D_{2j-1}} H_{\tilde{u}}(\tilde{u}_j) \ , \ a = 1, \dots, d_{2j-1} \ . \tag{7.3.7}$$

If $r = 2k$ and $d_{2k} \geq 1$, we set

$$V\left(\frac{z^b}{D_{2k}} u'_k\right) = F(\sigma_k^2) \frac{z^b}{D_{2k}} \tilde{u}'_k \ , \ b = 0, \dots, d_{2k} \ . \tag{7.3.8}$$

Similarly, if $0 \in \Sigma_K$, we define $V(u'_0) = F(0)\tilde{u}'_0$. Using (7.3.8) we get $V(u'_k) = F(\sigma_k^2)\tilde{u}'_k$. Consequently,

$$V(u) = V(u'_0) + \sum_k V(u'_k) = F(K_{\tilde{u}}^2)\tilde{u} \ .$$

A similar argument combined with Proposition 3 leads to

$$VK_u = K_{\tilde{u}} V.$$

Using again Lemma 22, V is unitary from the closed range of H_u onto the closed range of $H_{\tilde{u}}$.

Now we define U and V on the kernel of H_u which is either $\{0\}$ or an infinite dimensional separable Hilbert space. From Corollary 3, the cancellation of $\ker H_u$ only depends on the s_r's. Therefore, $\ker H_u$ and $\ker H_{\tilde{u}}$ are isometric. We then define $U = V$ from $\ker H_u$ onto $\ker H_{\tilde{u}}$ to be any unitary operator.

It remains to prove that $U^*V = \overline{F}(H_u^2)F(K_u^2)$. On $\ker H_u$, it is trivial since $U^*V = I = \overline{F}(0)F(0)$. Similarly, it is trivial on vectors

$$\frac{z^a}{D_{2k}} u'_k \ , a = 1, \dots, d_{2k} \ .$$

It remains to prove the equality for u'_0, u'_k. We write

$$\begin{aligned} U^*V(u'_k) &= F(\sigma_k^2)U^*(\tilde{u}'_k) = F(\sigma_k^2)U^*\left(\kappa_k^2\sum_j \frac{\tilde{u}_j}{\rho_j^2-\sigma_k^2}\right) \\ &= F(\sigma_k^2)\sum_j \overline{F}(\rho_j^2)\kappa_k^2\frac{u_j}{\rho_j^2-\sigma_k^2} = F(\sigma_k^2)\overline{F}(H_u^2)u'_k \\ &= \overline{F}(H_u^2)F(K_u^2)(u'_k)\ . \end{aligned}$$

A similar arguments holds for $U^*V(u'_0)$.

It remains to prove Lemma 22. It is sufficient to prove that the only solution of the homogeneous system

$$\left\{\begin{array}{l} \zeta_{d-b}+a_1\zeta_{d-b-1}+\dots+a_d\zeta_{-b}=0\ ,\ \ b=0,\dots,d-1\ , \\ \zeta_0+a_1\zeta_{-1}+\dots+a_d\zeta_{-d}=0\ , \end{array}\right. \tag{7.3.9}$$

with $\overline{\zeta}_k=\zeta_{-k}$, $k=0,\dots,d$, is the trivial solution $\zeta=0$.

We proceed by induction on d. For $d=1$, the system reads

$$\left\{\begin{array}{l} \zeta_1+a_1\zeta_0=0\ ,\ \ , \\ \zeta_0+a_1\overline{\zeta}_1=0\ . \end{array}\right.$$

Since $|a_1|<1$, this trivially implies $\zeta_0=\zeta_1=0$.

For a general d, we plug the expression

$$\zeta_d=-(a_1\zeta_{d-1}+\dots+a_d\zeta_0)$$

into the last equation. We get

$$\zeta_0+b_1\overline{\zeta}_1+\dots+b_{d-1}\overline{\zeta}_{d-1}=0 \tag{7.3.10}$$

with

$$b_k=\frac{a_k-a_d\overline{a}_{d-k}}{1-|a_d|^2}\ ,k=1,\dots,d-1\ .$$

Notice that from Proposition 2, $|a_d|<1$ and the polynomial $z^{d-1}+b_1z^{d-2}+\dots+b_{d-1}$ has all its roots in $\mathbb{D}$. For $b=1,\dots,d-1$, we multiply by a_d the conjugate of equation

$$\zeta_b+a_1\zeta_{b-1}+\dots+a_d\zeta_{b-d}=0$$

and substract the result from equation

$$\zeta_{d-b}+a_1\zeta_{d-b-1}+\dots+a_d\zeta_{-b}=0\ .$$

This yields

$$\zeta_{d-b} + b_1\zeta_{d-b-1} + \cdots + b_{d-1}\zeta_{1-b} = 0 \ .$$

Together with Equation (7.3.10), this is exactly the system at order $d-1$ with coefficients $b_1, \ldots, b_{d-1}$. By induction, we obtain

$$\zeta_0 = \zeta_1 = \cdots = \zeta_{d-1} = 0$$

and finally $\zeta_d = 0$.

This completes the proof.

□

CHAPTER 8

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